

# INTEGRAL STRUCTURES OF MULTIPLE POLYLOGARITHMS AT ROOTS OF UNITY

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**Abstract.** In this paper, for any positive integer  $N$  we shall study the special values of multiple polylogarithms at  $N$ th roots of unity, called multiple polylogarithmic values (MPVs) of level  $N$ . By standard conjectures linear relations exist only between MPVs of the same weight. Let  $\mathcal{MPV}_{\mathbb{Z}}(w, N)$  be the  $\mathbb{Z}$ -module spanned by MPVs of weight  $w$  and level  $N$ . Our main interest is to investigate for what  $w$  and  $N$  there exists a basis consisting of MPVs in  $\mathcal{MPV}_{\mathbb{Z}}(w, N)$ . In the scope of our investigation this problem always seems to have affirmative answers except for multiple zeta values (level one MPVs) of weight 6 and 7, provided that the conjectural dimensions of  $\mathcal{MPV}_{\mathbb{Q}}(w, N)$  are correct.

## 1. INTRODUCTION

For any positive integers  $s_1, \dots, s_\ell$ , the multiple polylogs of complex variables are defined as follows:

$$(1) \quad Li_{s_1, \dots, s_\ell}(x_1, \dots, x_\ell) = \sum_{k_1 > \dots > k_\ell > 0} \frac{x_1^{k_1} \cdots x_\ell^{k_\ell}}{k_1^{s_1} \cdots k_\ell^{s_\ell}}.$$

Conventionally one refers  $l$  as the *depth* and  $wt(s_1, \dots, s_\ell) := s_1 + \dots + s_\ell$  as the *weight*. For any positive integer  $N$  (called the *level*) we shall study the multiple polylogarithms values at  $N$ th roots of unity (MPVs):

$$(2) \quad L_N(s_1, \dots, s_\ell | i_1, \dots, i_\ell) := Li_{s_1, \dots, s_\ell}(\mu^{i_1}, \dots, \mu^{i_\ell}),$$

where  $\mu = \mu_N := \exp(2\pi i/N)$ . It is well-known that standard conjectures in arithmetic geometry implies that  $\mathbb{Q}$ -linear relations only exist between MPVs of the same weight. Many of these relations can be derived from the following four families of standard relations (see [10, 11]): regularized double shuffle relations, regularized distribution relations, lifted versions of such relations from lower weights, and seeded relations which are produced by relations of weight one MPVs.

The structure of MPVs over  $\mathbb{Q}$  is intimately related to a number of deep questions in arithmetic algebraic geometry and quantum field theory, see for e.g. [3, 4, 6, 7]. However, by intensive MAPLE computation, some integral structures seem to emerge. In this paper we will present this and prove some results related to the following

**Main Problem.** Let  $\mathcal{MPV}_{\mathbb{Z}}(w, N)$  be the  $\mathbb{Z}$ -module generated by MPVs of weight  $w$  and level  $N$ . If  $(w, N) \neq (6, 1), (7, 1)$  is there always a MPV basis of  $\mathcal{MPV}_{\mathbb{Z}}(w, N)$ ?

Let  $\mathcal{MPV}_{\mathbb{Q}}(w, N)$  be the  $\mathbb{Q}$ -span of all MPVs of weight  $w$  and level  $N$  and  $d(w, N)$  be its dimension. In [6] Deligne proves that if a variant of Grothendieck's period conjecture is true then  $d(w, 2) = F_w$  are given by Fibonacci numbers ( $F_1 = 1, F_2 = 2, \dots$ ),  $d(w, 3) = d(w, 4) = 2^w$ , and  $d(w, 8) = 3^w$ . Assuming the same conjecture we show in [10] that when  $N = p$  is a prime  $\geq 5$  then the bound of  $d(2, p) \leq \frac{(5p+7)(p+1)}{24}$  produced by the standard relations are sharp (see Thm. 5.1 of op. cit.)

In order to study the integral structure of  $\mathcal{MPV}_{\mathbb{Z}}(w, N)$  we need to provide explicit generators which is not always feasible at present. The difficulty is that when weight  $w \geq 2$  if  $N$  is a non-standard level in the sense of [11] (i.e.,  $N$  has at least two distinct prime factors or  $N = 2^r$  or  $N = 3^r$  with  $r \geq 2$ ) then often there are hidden relations besides the standard ones. From numerical evidence it is very likely that the only exceptions to this incompleteness of standard relations are when  $(w, N) = (2, 4), (2, 6)$  and  $(2, 9)$ . For example, when  $N = 4$  we show [10] that the upper bounds produced by the standard relations  $d(3, 4) \leq 9$  and  $d(4, 4) \leq 21$  are not optimal according to [7, 5.25]: in weight three at least one relation is missing, and in weight four at least five relations are missing. Fortunately for us in these two cases we can use octahedral symmetry of  $\mathbb{P}^1 - (\{0, \infty\} \cup \mu_4)$  to produce all the missing relations (see [10]).

Most of the MPV identities in this paper are discovered with the help of MAPLE using symbolic computations. We then verified every relation by GiNaC [9] with an error  $< 10^{-90}$ .

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## 2. MULTIPLE ZETA VALUES

Recall that multiple zeta values (MZVs for short) are simply level one MPVs. All the relations in this section are well-known to the experts. However, it seems that most of the relations never appeared in this particular form in print before. In what follows we will assume Zagier's conjecture on the dimension of MZVs:  $d(1, 1) = 0$ ,  $d(2, 1) = 1$ ,  $d(3, 1) = 1$  and  $d(w, 1) = d(w - 2, 1) + d(w - 3, 1)$  for all  $w \geq 4$ . In weight three Euler showed that

$$\zeta(3) = \zeta(2, 1).$$

In weight four:

$$\zeta(4) = 4\zeta(3, 1), \quad \zeta(2, 1, 1) = 4\zeta(3, 1), \quad \zeta(2, 2) = 3\zeta(3, 1).$$

In weight five:

$$\begin{aligned}\zeta(5) &= \zeta(2, 1, 1, 1) = 2\zeta(2, 2, 1) + 6\zeta(3, 1, 1), & \zeta(4, 1) &= \zeta(3, 1, 1), \\ \zeta(2, 3) &= \zeta(2, 1, 2) = \zeta(2, 2, 1) + 5\zeta(3, 1, 1), & \zeta(3, 2) &= \zeta(2, 2, 1).\end{aligned}$$

In weight six:  $\zeta(3, 1, 1, 1) = \zeta(5, 1)$ ,  $\zeta(2, 3, 1) = \zeta(3, 1, 2)$ , and

$$\begin{aligned}\zeta(6) &= \zeta(2, 1, 1, 1, 1) = 72\zeta(5, 1) - 24\zeta(3, 1, 2), \\ \zeta(4, 2) &= \zeta(2, 2, 1, 1) = 10\zeta(5, 1) - 4\zeta(3, 1, 2), \\ \zeta(3, 3) &= \zeta(2, 1, 2, 1) = 17\zeta(5, 1) - 6\zeta(3, 1, 2), \\ \zeta(2, 4) &= \zeta(2, 1, 1, 2) = 44\zeta(5, 1) - 14\zeta(3, 1, 2), \\ \zeta(4, 1, 1) &= -\frac{5}{2}\zeta(5, 1) + \frac{3}{2}\zeta(3, 1, 2), \\ \zeta(3, 2, 1) &= \frac{27}{2}\zeta(5, 1) - \frac{13}{2}\zeta(3, 1, 2), \\ \zeta(2, 2, 2) &= \frac{27}{2}\zeta(5, 1) - \frac{9}{2}\zeta(3, 1, 2), \\ \zeta(2, 1, 3) &= \frac{95}{2}\zeta(5, 1) - \frac{33}{2}\zeta(3, 1, 2).\end{aligned}$$

By considering the fractional coefficients it is easy to show that our Main Problem has negative answer in this case. To see this, let  $A = \zeta(5, 1)$  and  $B = \zeta(3, 1, 2)$  and

$$[\zeta(6) \ \zeta(4, 2) \ \zeta(3, 3) \ \zeta(2, 4) \ \zeta(4, 1, 1) \ \zeta(3, 2, 1) \ \zeta(2, 2, 2) \ \zeta(2, 1, 3)] = [A \ B]M$$

where

$$M = \begin{bmatrix} 72 & 10 & 17 & 44 & -\frac{5}{2} & \frac{27}{2} & \frac{27}{2} & \frac{95}{2} \\ -24 & -4 & -6 & -14 & \frac{3}{2} & -\frac{13}{2} & -\frac{9}{2} & -\frac{33}{2} \end{bmatrix}.$$

If another pair, say  $[C \ D] = [A \ B]T$  (for some  $2 \times 2$  minor  $T$  of  $M$ ) form a  $\mathbb{Z}$  basis then not only  $T^{-1}M$  is an integral matrix but also  $T^{-1} \in M_2(\mathbb{Z})$  since both entries in  $[A \ B] = [C \ D]T^{-1}$  are  $\mathbb{Z}$ -linear combinations of  $C$  and  $D$ . Let  $T_1 = 2T$  and

$$M_1 = 2M = \begin{bmatrix} 144 & 20 & 34 & 88 & -5 & 27 & 27 & 95 \\ -48 & -8 & -12 & -28 & 3 & -13 & -9 & -33 \end{bmatrix}.$$

Then  $T_1 \in M_2(\mathbb{Z})$  is a  $2 \times 2$  minor of  $M_1$ ,  $T_1^{-1} \in (1/2)M_2(\mathbb{Z})$ , and  $M_1 \in M_2(\mathbb{Z})$ . We claim that all of the entries of  $T_1$  are odd numbers. Indeed, if all entries of  $T_1$  are even then it is readily to see that  $\det(T_1) \equiv 0 \pmod{8}$  so  $T_1^{-1} \notin (1/2)M_2(\mathbb{Z})$ . If only one column of  $T_1$  has odd entries and the other column  $\neq \begin{bmatrix} 34 \\ -12 \end{bmatrix}$  then it is obvious that  $\det(T_1) \equiv 0 \pmod{4}$  so  $T_1^{-1} \notin (1/2)M_2(\mathbb{Z})$ . If one column has odd entries and the other column  $= \begin{bmatrix} 34 \\ -12 \end{bmatrix}$  then a quick computation shows that again  $T_1^{-1} \notin (1/2)M_2(\mathbb{Z})$ . Consequently, all of the entries of  $T_1$  are odd numbers in which case computation shows that  $\det(T_1) \equiv 0 \pmod{4}$  (all odd entries are  $\equiv -1 \pmod{4}$ ) and therefore  $T_1^{-1} \notin (1/2)M_2(\mathbb{Z})$ . This contradiction implies that there don't exist two MZVs of weight six such that every MZV of weight six is a  $\mathbb{Z}$ -linear combination of these two.

In weight seven let  $A = \zeta(6, 1)/2$  (note the coefficient  $1/2$ ),  $B = \zeta(5, 1, 1)$  and  $C = \zeta(4, 2, 1)$ . Then  $\zeta(3, 1, 1, 1, 1) = 2A$ ,  $\zeta(4, 1, 1, 1) = B$ ,  $\zeta(3, 2, 1, 1) = C$ , and

$$\begin{aligned}
\zeta(7) &= \zeta(2, 1, 1, 1, 1, 1) = 16A + 128B + 48C, \\
\zeta(5, 2) &= \zeta(2, 2, 1, 1, 1) = 6A - 4B, \\
\zeta(4, 3) &= \zeta(2, 1, 2, 1, 1) = -28A + 56B + 16C, \\
\zeta(3, 4) &= \zeta(2, 1, 1, 2, 1) = 28A - 12B, \\
\zeta(2, 5) &= \zeta(2, 1, 1, 1, 2) = 8A + 88B + 32C, \\
\zeta(4, 1, 2) &= \zeta(2, 3, 1, 1) = 25A - 35B - 10C, \\
\zeta(3, 3, 1) &= \zeta(3, 1, 2, 1) = -13A + 22B + 6C, \\
\zeta(3, 2, 2) &= \zeta(2, 2, 2, 1) = -32A + 53B + 15C, \\
\zeta(3, 1, 3) &= \zeta(2, 1, 3, 1) = 11B + 4C, \\
\zeta(2, 4, 1) &= \zeta(3, 1, 1, 2) = 21A - 27B - 7C, \\
\zeta(2, 3, 2) &= \zeta(2, 2, 1, 2) = -15A + 34B + 10C, \\
\zeta(2, 2, 3) &= \zeta(2, 1, 2, 2) = 45A - 42B - 9C, \\
\zeta(2, 1, 4) &= \zeta(2, 1, 1, 3) = -15A + 111B + 38C.
\end{aligned}$$

By similar argument as weight six we see that our Main Problem has negative answer in the weight seven case too. But the answer is affirmative in weight eight as shown below: let  $A = \zeta(5, 1, 1, 1)$ ,  $B = \zeta(2, 1, 1, 3, 1)$ ,  $C = \zeta(3, 1, 1, 1, 2)$ , and  $D = \zeta(2, 4, 1, 1)$ , then

$$\begin{aligned}
\zeta(8) &= 20160A + 2304B + 8064D - 13824C, \\
\zeta(7, 1) &= 2978A + 338B - 2040C + 1190D, \\
\zeta(6, 2) &= -6208A - 704B + 4252C - 2480D, \\
\zeta(5, 3) &= 3150A + 358B - 2158C + 1258D, \\
\zeta(4, 4) &= 1680A + 192B - 1152C + 672D, \\
\zeta(3, 5) &= -1088A - 120B + 742C - 432D, \\
\zeta(2, 6) &= 19648A + 2240B - 13468C + 7856D.
\end{aligned}$$

In depth three:  $\zeta(3, 1, 4) = B$ ,  $\zeta(2, 5, 1) = C$ , and

$$\begin{aligned}
\zeta(6, 1, 1) &= 2821A + 320B - 1932C + 1127D, \\
\zeta(5, 2, 1) &= -10041A - 1139B + 6877C - 4011D, \\
\zeta(5, 1, 2) &= 10866A + 1233B - 7444C + 4342D, \\
\zeta(4, 3, 1) &= 6038A + 685B - 4135C + 2410D, \\
\zeta(4, 2, 2) &= -7014A - 796B + 4806C - 2802D, \\
\zeta(4, 1, 3) &= -698A - 79B + 478C - 278D, \\
\zeta(3, 4, 1) &= -2048A - 232B + 1401C - 816D, \\
\zeta(3, 3, 2) &= -7119A - 808B + 4879C - 2845D, \\
\zeta(3, 2, 3) &= 8795A + 999B - 6028C + 3513D, \\
\zeta(2, 4, 2) &= 10250A + 1164B - 7024C + 4094D, \\
\zeta(2, 3, 3) &= -2291A - 259B + 1569C - 913D, \\
\zeta(2, 2, 4) &= 964A + 112B - 662C + 388D, \\
\zeta(2, 1, 5) &= 9637A + 1103B - 6610C + 3855D.
\end{aligned}$$

In depth four:  $\zeta(4, 1, 1, 2) = D$  and

$$\begin{aligned}
 \zeta(4, 2, 1, 1) &= -1395A - 158B + 954C - 556D, \\
 \zeta(4, 1, 2, 1) &= 5040A + 572B - 3453C + 2013D, \\
 \zeta(3, 3, 1, 1) &= 5040A + 572B - 3453C + 2013D, \\
 \zeta(3, 2, 2, 1) &= -17073A - 1938B + 11700C - 6822D, \\
 \zeta(3, 2, 1, 2) &= 2411A + 274B - 1653C + 962D, \\
 \zeta(3, 1, 3, 1) &= 105A + 12B - 72C + 42D, \\
 \zeta(3, 1, 2, 2) &= 8799A + 999B - 6030C + 3516D, \\
 \zeta(3, 1, 1, 3) &= -3008A - 341B + 2061C - 1201D, \\
 \zeta(2, 3, 2, 1) &= 2411A + 274B - 1653C + 962D, \\
 \zeta(2, 3, 1, 2) &= -4297A - 488B + 2946C - 1714D, \\
 \zeta(2, 2, 3, 1) &= 8799A + 999B - 6030C + 3516D, \\
 \zeta(2, 2, 2, 2) &= 525A + 60B - 360C + 210D, \\
 \zeta(2, 2, 1, 3) &= 2201A + 251B - 1509C + 878D, \\
 \zeta(2, 1, 4, 1) &= -3008A - 341B + 2061C - 1201D, \\
 \zeta(2, 1, 3, 2) &= 2201A + 251B - 1509C + 878D, \\
 \zeta(2, 1, 2, 3) &= -4057A - 458B + 2778C - 1616D, \\
 \zeta(2, 1, 1, 4) &= 15465A + 1764B - 10602C + 6182D.
 \end{aligned}$$

In depth five:

$$\begin{aligned}
 \zeta(4, 1, 1, 1, 1) &= 2821A + 320B - 1932C + 1127D, \\
 \zeta(3, 2, 1, 1, 1) &= -10041A - 1139B + 6877C - 4011D, \\
 \zeta(3, 1, 2, 1, 1) &= 6038A + 685B - 4135C + 2410D, \\
 \zeta(2, 2, 2, 1, 1) &= -7014A - 796B + 4806C - 2802D, \\
 \zeta(3, 1, 1, 2, 1) &= -2048A - 232B + 1401C - 816D, \\
 \zeta(2, 3, 1, 1, 1) &= 10866A + 1233B - 7444C + 4342D, \\
 \zeta(2, 2, 1, 2, 1) &= -7119A - 808B + 4879C - 2845D, \\
 \zeta(2, 2, 1, 1, 2) &= 10250A + 1164B - 7024C + 4094D, \\
 \zeta(2, 1, 3, 1, 1) &= -698A - 79B + 478C - 278D, \\
 \zeta(2, 1, 2, 2, 1) &= 8795A + 999B - 6028C + 3513D, \\
 \zeta(2, 1, 2, 1, 2) &= -2291A - 259B + 1569C - 913D, \\
 \zeta(2, 1, 1, 2, 2) &= 964A + 112B - 662C + 388D, \\
 \zeta(2, 1, 1, 1, 3) &= 9637A + 1103B - 6610C + 3855D.
 \end{aligned}$$

Finally, in depth six or seven:

$$\begin{aligned}
 \zeta(3, 1, 1, 1, 1, 1) &= 2978A + 338B - 2040C + 1190D, \\
 \zeta(2, 2, 1, 1, 1, 1) &= -6208A - 704B + 4252C - 2480D, \\
 \zeta(2, 1, 1, 1, 2, 1) &= -1088A - 120B + 742C - 432D, \\
 \zeta(2, 1, 2, 1, 1, 1) &= 3150A + 358B - 2158C + 1258D, \\
 \zeta(2, 1, 1, 2, 1, 1) &= 1680A + 192B - 1152C + 672D, \\
 \zeta(2, 1, 1, 1, 1, 2) &= 19648A + 2240B - 13468C + 7856D, \\
 \zeta(2, 1, 1, 1, 1, 1, 1) &= 20160A + 2304B - 13824C + 8064D.
 \end{aligned}$$

## 3. ALTERNATING EULER SUMS (NAMELY, LEVEL TWO MPVs)

MPVs of level two are often called (alternating) Euler sums and are denoted by

$$\zeta(s_1, \dots, s_\ell; x_1, \dots, x_\ell) := Li_{s_1, \dots, s_\ell}(x_1, \dots, x_\ell)$$

where  $x_j = \pm 1$ . To save space, conventionally we use  $\bar{s}_j$  to signify that  $x_j = -1$ . For example, there is only one Euler sum with weight one:  $\zeta(\bar{1}) = -\log 2$ .

Broadhurst conjectures that the dimension of  $\mathbb{Q}$ -vector space generated by weight  $w$  Euler sum sums is given by the Fibonacci numbers:

$$(3) \quad d(1, 2) = 1, d(2, 2) = 2, d(n, 2) = d(n-1, 2) + d(n-2, 2) \quad \forall n \geq 3.$$

Deligne and Goncharov [7, 5.25] proved that these numbers provide the upper bounds. More recently, Deligne showed that Broadhurst's conjecture is true if a variant of Grothendieck's period conjecture holds, with the following basis in weight  $w$ :

$$\left\{ (2\pi i)^{2p} \prod_{\vec{s}} \zeta(\bar{s}_1, s_2, \dots, s_\ell) : \vec{s} \in \mathfrak{L}, 2p + \sum_{\vec{s}} wt(\vec{s}) = w \right\},$$

where  $\mathfrak{L}$  is the set of the Lyndon words of odd numbers with the following order:  $\dots \prec 5 \prec 3 \prec 1$ . In [13] Zlobin proposed some other possible basis:  $\zeta(\bar{s}_1, s_2, \dots, s_\ell)$ , where  $s_j \in \{1, 2\}$  and  $wt(s_1, s_2, \dots, s_\ell) = w$ .

In this section we assume Broadhurst conjecture (3) is true and provide explicitly integral bases for  $\mathcal{MPV}_{\mathbb{Z}}(w, 2)$  when the weight  $w < 6$ . We can also verify that the basis of Deligne and Zlobin are both valid for the  $\mathbb{Q}$ -vector space  $\mathcal{MPV}_{\mathbb{Q}}(w, 2)$ , but in general neither is good for the integral  $\mathcal{MPV}_{\mathbb{Z}}(w, 2)$ .

**Proposition 3.1.** *All the weight two Euler sums can be expressed as  $\mathbb{Z}$ -linear combinations of  $\zeta(\bar{2})$  and  $\zeta(\bar{1}, 1)$ :*

$$\zeta(2) = -2\zeta(\bar{2}), \quad \zeta(\bar{1}, \bar{1}) = \zeta(\bar{2}) + \zeta(\bar{1}, 1)$$

*Proof.* It is easy to see from regularized double shuffle relations (see [11, §8.2]) that

$$\zeta(2) = -2\zeta(\bar{1}, \bar{1}) + 2\zeta(\bar{1}, 1), \quad \zeta(\bar{2}) = -\zeta(\bar{1}, 1) + \zeta(\bar{1}, \bar{1}).$$

□

*Remark 3.2.* From the proposition and a stuffle relation we get

$$2\zeta(\bar{1}, 1) = 2\zeta(\bar{1}, \bar{1}) - 2\zeta(\bar{2}) = \zeta(\bar{1})^2 = \log^2 2.$$

Hence it is clear that Deligne's basis is good for  $\mathcal{MPV}_{\mathbb{Q}}(2, 2)$  and Zlobin's is even good for  $\mathcal{MPV}_{\mathbb{Z}}(2, 2)$ . But we don't know how to prove that  $\pi^2$  and  $\log^2 2$  are linearly independent over  $\mathbb{Q}$  (which can be deduced from a variant of Grothendieck's period conjecture).

**Proposition 3.3.** *We can express all weight three Euler sums as  $\mathbb{Z}$ -linear combinations of  $\zeta(\bar{2}, 1)$ ,  $\zeta(\bar{1}, 1, 1)$  and  $\zeta(\bar{1}, 2)$ :*

$$\begin{aligned}
 \zeta(3) &= 8\zeta(\bar{2}, 1) , \\
 \zeta(\bar{3}) &= -6\zeta(\bar{2}, 1) , \\
 \zeta(2, 1) &= 8\zeta(\bar{2}, 1) , \\
 \zeta(2, \bar{1}) &= 2\zeta(\bar{2}, 1) - 3\zeta(\bar{1}, 2) , \\
 \zeta(\bar{2}, \bar{1}) &= 3\zeta(\bar{1}, 2) - 7\zeta(\bar{2}, 1) , \\
 \zeta(\bar{1}, \bar{2}) &= -2\zeta(\bar{1}, 2) + \zeta(\bar{2}, 1) , \\
 \zeta(\bar{1}, 1, \bar{1}) &= \zeta(\bar{2}, 1) + \zeta(\bar{1}, 1, 1) , \\
 \zeta(\bar{1}, \bar{1}, 1) &= \zeta(\bar{1}, 2) - 5\zeta(\bar{2}, 1) + \zeta(\bar{1}, 1, 1) , \\
 \zeta(\bar{1}, \bar{1}, \bar{1}) &= \zeta(\bar{1}, 2) + \zeta(\bar{1}, 1, 1) .
 \end{aligned}$$

*Proof.* When weight is three, by double shuffle relations we have

$$\begin{aligned}
 \zeta(\bar{1}, 1, \bar{1}) + 2\zeta(\bar{1}, \bar{1}, 1) + \zeta(\bar{1}, \bar{2}) + \zeta(2, 1) - 3\zeta(\bar{1}, 1, 1) &= 0, \\
 2\zeta(\bar{1}, \bar{1}, \bar{1}) + \zeta(\bar{1}, 2) + \zeta(2, \bar{1}) - 2\zeta(\bar{1}, 1, \bar{1}) &= 0, \\
 \zeta(\bar{2}, \bar{1}) + \zeta(\bar{1}, \bar{2}) + \zeta(3) - 2\zeta(\bar{2}, 1) - \zeta(\bar{1}, 2) &= 0, \\
 \zeta(\bar{1}, 2) + \zeta(\bar{3}) - \zeta(\bar{2}, \bar{1}) - \zeta(\bar{1}, \bar{2}) &= 0.
 \end{aligned}$$

These are far from enough to prove the proposition. But by regularized double shuffle relations we have five more relations:

$$\begin{aligned}
 \zeta(\bar{3}) + 2\zeta(\bar{2}, 1) + \zeta(\bar{1}, 2) + 2\zeta(\bar{1}, 1, 1) - \zeta(2, \bar{1}) + \zeta(\bar{1})\zeta(2) - 2\zeta(\bar{1}, \bar{1}, 1) &= 0, \\
 \zeta(\bar{1}, 1, \bar{1}) - \zeta(\bar{2}, 1) - \zeta(\bar{1}, 2) - 2\zeta(\bar{1}, 1, 1) + \zeta(\bar{1}, \bar{1}, \bar{1}) &= 0, \\
 \zeta(\bar{1}, \bar{1}, 1) - \zeta(\bar{2}, \bar{1}) - \zeta(\bar{1}, \bar{2}) - \zeta(\bar{1}, 1, \bar{1}) &= 0, \\
 \zeta(\bar{2}, \bar{1}) - \zeta(\bar{3}) - \zeta(\bar{2}, 1) + \zeta(2, \bar{1}) &= 0, \\
 \zeta(2, 1) - \zeta(3) &= 0.
 \end{aligned}$$

Now the proposition follows from the stuffle relation:  $\zeta(\bar{1})\zeta(2) = \zeta(\bar{3}) + \zeta(2, \bar{1}) + \zeta(\bar{1}, 2)$ .  $\square$

*Remark 3.4.* It is generally believed that there should be no further linear relations among  $\zeta(\bar{2}, 1)$ ,  $\zeta(\bar{1}, 1, 1)$  and  $\zeta(\bar{1}, 2)$  which gives  $EZ_3 = 3$ . This is easy to see to be equivalence to the linear independence of  $\zeta(3)$ ,  $\zeta(\bar{1})\zeta(2)$  and  $\zeta(\bar{1}, \bar{1}, 1)$ .

**Proposition 3.5.** *All weight four Euler sums are  $\mathbb{Z}$ -linear combinations of  $A = \zeta(\bar{2}, 1, 1)$ ,  $B = \zeta(\bar{2}, 2)$ ,  $C = \zeta(\bar{1}, 2, 1)$ ,  $D = \zeta(\bar{1}, 1, 2)$ , and  $E = \zeta(\bar{1}, 1, 1, 1)$ . For length one and*

two:

$$\begin{aligned}
\zeta(4) &= 64A + 16B, \\
\zeta(\bar{4}) &= -56A - 14B, \\
\zeta(3, 1) &= 16A + 4B, \\
\zeta(3, \bar{1}) &= 118A + 19B + 14C, \\
\zeta(2, 2) &= 48A + 12B, \\
\zeta(\bar{3}, 1) &= 10A + 2B, \\
\zeta(\bar{3}, \bar{1}) &= -140A - 24B - 14C, \\
\zeta(2, \bar{2}) &= -24A - 7B, \\
\zeta(\bar{2}, \bar{2}) &= -12A - 3B, \\
\zeta(\bar{1}, 3) &= -38A - 5B - 6C, \\
\zeta(\bar{1}, \bar{3}) &= 58A + 8B + 8C.
\end{aligned}$$

For length three:

$$\begin{aligned}
\zeta(2, 1, 1) &= 64A + 16B, \\
\zeta(2, 1, \bar{1}) &= 16A + 2B + 6C + 3D, \\
\zeta(2, \bar{1}, 1) &= 22A + 3B + C - 3D, \\
\zeta(2, \bar{1}, \bar{1}) &= 100A + 13B + 9C - 6D, \\
\zeta(\bar{2}, 1, \bar{1}) &= 91A + 14B + 8C - 3D, \\
\zeta(\bar{2}, \bar{1}, 1) &= -161A - 26B - 15C + 3D, \\
\zeta(\bar{2}, \bar{1}, \bar{1}) &= -101A - 14B - 9C + 6D, \\
\zeta(\bar{1}, 2, \bar{1}) &= -102A - 14B - 8C + 6D, \\
\zeta(\bar{1}, \bar{2}, 1) &= 69A + 11B + 8C, \\
\zeta(\bar{1}, \bar{2}, \bar{1}) &= 63A + 8B + 3C - 6D, \\
\zeta(\bar{1}, 1, \bar{2}) &= 21A + 3B + C - 2D, \\
\zeta(\bar{1}, \bar{1}, \bar{2}) &= A + 2B + D.
\end{aligned}$$

For length four,

$$\begin{aligned}
\zeta(\bar{1}, 1, 1, \bar{1}) &= A + E, \\
\zeta(\bar{1}, 1, \bar{1}, 1) &= 11A + 2B + C + E, \\
\zeta(\bar{1}, 1, \bar{1}, \bar{1}) &= C + E, \\
\zeta(\bar{1}, \bar{1}, 1, 1) &= -83A - 16B - 5C + D + E, \\
\zeta(\bar{1}, \bar{1}, 1, \bar{1}) &= -38A - 5B - 5C + D + E, \\
\zeta(\bar{1}, \bar{1}, \bar{1}, 1) &= D + E, \\
\zeta(\bar{1}, \bar{1}, \bar{1}, \bar{1}) &= A + B + D + E.
\end{aligned}$$

The next proposition shows that the  $\mathbb{Q}$ -basis conjectured by Zlobin can not be chosen as the  $\mathbb{Z}$ -linear basis in general.

**Proposition 3.6.** *All weight five Euler sums are  $\mathbb{Q}$ -linear combinations of  $\zeta(\bar{1}, 1, 1, 1, 1)$ ,  $\zeta(\bar{1}, 1, 2, 1)$ ,  $\zeta(\bar{2}, 1, 1, 1)$ ,  $\zeta(\bar{1}, 1, 1, 2)$ ,  $\zeta(\bar{1}, 2, 1, 1)$ ,  $\zeta(\bar{2}, 1, 2)$ ,  $\zeta(\bar{2}, 2, 1)$  and  $\zeta(\bar{1}, 2, 2)$ . For example*

$$\zeta(3, 1, 1) = -\frac{448}{39}\zeta(\bar{2}, 1, 1, 1) - \frac{112}{39}\zeta(\bar{2}, 2, 1) - \frac{48}{13}\zeta(\bar{2}, 1, 2).$$

Furthermore, all weight five Euler sums are  $\mathbb{Z}$ -linear combinations of

$$\begin{aligned} A &= \zeta(\bar{1}, \bar{1}, \bar{1}, 2), & B &= \zeta(\bar{2}, 1, \bar{1}, \bar{1}), & C &= \zeta(\bar{1}, 1, \bar{1}, \bar{2}), & D &= \zeta(\bar{2}, 1, 1, 1), \\ E &= \zeta(\bar{1}, \bar{1}, \bar{1}, 1, 1), & F &= \zeta(2, 2, \bar{1}), & G &= \zeta(\bar{1}, 1, \bar{1}, 1, \bar{1}), & H &= \zeta(\bar{1}, \bar{1}, \bar{1}, \bar{1}, \bar{1}). \end{aligned}$$

For length one and two:

$$\begin{aligned} \zeta(5) &= -13504A + 1856B - 1344C + 26880D - 18752E - 640F - 31552G + 50304H, \\ \zeta(\bar{5}) &= 12660A - 1740B + 1260C - 25200D + 17580E + 600F + 29580G - 47160H, \\ \zeta(4, 1) &= -9808A + 1344B - 944C + 19632D - 13648E - 464F - 22848G + 36496H, \\ \zeta(4, \bar{1}) &= -14918A + 2044B - 1434C + 29862D - 20758E - 704F - 34748G + 55506H, \\ \zeta(\bar{4}, 1) &= 3638A - 498B + 346C - 7296D + 5066E + 172F + 8466G - 13532H, \\ \zeta(\bar{4}, \bar{1}) &= 19862A - 2722B + 1914C - 39744D + 27634E + 938F + 46274G - 73908H, \\ \zeta(3, 2) &= 22672A - 3104B + 2160C - 45456D + 31568E + 1072F + 52768G - 84336H, \\ \zeta(3, \bar{2}) &= 4562A - 626B + 446C - 9108D + 6342E + 216F + 10642G - 16984H, \\ \zeta(\bar{3}, 2) &= -6552A + 898B - 632C + 13110D - 9116E - 310F - 15266G + 24382H, \\ \zeta(\bar{3}, \bar{2}) &= -17848A + 2444B - 1704C + 35772D - 24848E - 844F - 41548G + 66396H, \\ \zeta(2, 3) &= -26368A + 3616B - 2560C + 52704D - 36672E - 1248F - 61472G + 98144H, \\ \zeta(2, \bar{3}) &= 6792A - 934B + 680C - 13506D + 9428E + 322F + 15878G - 25306H, \\ \zeta(\bar{2}, \bar{3}) &= 24902A - 3412B + 2394C - 49854D + 34654E + 1178F + 58004G - 92658H, \\ \zeta(\bar{2}, 3) &= -8622A + 1182B - 834C + 17244D - 11994E - 408F - 20094G + 32088H, \\ \zeta(\bar{1}, 4) &= 5266A - 720B + 494C - 10582D + 7338E + 248F + 12240G - 19578H, \\ \zeta(\bar{1}, \bar{4}) &= -8990A + 1230B - 850C + 18044D - 12522E - 424F - 20910G + 33432H. \end{aligned}$$

*For length three,*

$$\begin{aligned}
\zeta(3, 1, 1) &= -9808A + 1344B - 944C + 19632D - 13648E - 464F - 22848G + 36496H, \\
\zeta(3, 1, \bar{1}) &= -5314A + 725B - 500C + 10677D - 7402E - 250F - 12339G + 19741H, \\
\zeta(3, \bar{1}, 1) &= -2257A + 312B - 225C + 4489D - 3137E - 108F - 5290G + 8427H, \\
\zeta(3, \bar{1}, \bar{1}) &= -7299A + 1005B - 713C + 14566D - 10151E - 347F - 17057G + 27208H, \\
\zeta(\bar{3}, 1, 1) &= 4482A - 614B + 430C - 8974D + 6238E + 212F + 10438G - 16676H, \\
\zeta(\bar{3}, 1, \bar{1}) &= 9570A - 1308B + 908C - 19204D + 13328E + 452F + 22250G - 35578H, \\
\zeta(\bar{3}, \bar{1}, 1) &= 12462A - 1710B + 1204C - 24924D + 17338E + 590F + 29056G - 46394H, \\
\zeta(\bar{3}, \bar{1}, \bar{1}) &= -4288A + 582B - 396C + 8646D - 5978E - 201F - 9922G + 15900H, \\
\zeta(2, 2, 1) &= 22672A - 3104B + 2160C - 45456D + 31568E + 1072F + 52768G - 84336H, \\
\zeta(2, \bar{2}, 1) &= 3025A - 414B + 287C - 6065D + 4213E + 143F + 7038G - 11251H, \\
\zeta(2, \bar{2}, \bar{1}) &= 6421A - 881B + 627C - 12818D + 8927E + 303F + 14977G - 23904H, \\
\zeta(2, 1, 2) &= -26368A + 3616B - 2560C + 52704D - 36672E - 1248F - 61472G + 98144H, \\
\zeta(2, \bar{1}, 2) &= 2206A - 302B + 210C - 4428D + 3074E + 104F + 5134G - 8208H, \\
\zeta(2, 1, \bar{2}) &= 7958A - 1093B + 786C - 15861D + 11056E + 377F + 18581G - 29637H, \\
\zeta(2, \bar{1}, \bar{2}) &= 23513A - 3221B + 2255C - 47094D + 32727E + 1113F + 54757G - 87484H, \\
\zeta(\bar{2}, 2, 1) &= -12813A + 1755B - 1227C + 25664D - 17835E - 606F - 29835G + 47670H, \\
\zeta(\bar{2}, 2, \bar{1}) &= -20468A + 2804B - 1964C + 40988D - 28488E - 968F - 47668G + 76156H, \\
\zeta(\bar{2}, \bar{2}, 1) &= -5477A + 750B - 523C + 10977D - 7625E - 259F - 12750G + 20375H, \\
\zeta(\bar{2}, \bar{2}, \bar{1}) &= 12308A - 1686B + 1180C - 24654D + 17132E + 582F + 28662G - 45794H, \\
\zeta(\bar{2}, 1, 2) &= 12622A - 1729B + 1210C - 25281D + 17568E + 597F + 29393G - 46961H, \\
\zeta(\bar{2}, 1, \bar{2}) &= -3065A + 420B - 295C + 6135D - 4265E - 145F - 7140G + 11405H, \\
\zeta(\bar{2}, \bar{1}, 2) &= -14047A + 1923B - 1337C + 28170D - 19561E - 665F - 32691G + 52252H, \\
\zeta(\bar{2}, \bar{1}, \bar{2}) &= -9411A + 1290B - 909C + 18831D - 13095E - 445F - 21930G + 35025H, \\
\zeta(\bar{1}, 3, 1) &= 123A - 17B + 13C - 242D + 171E + 6F + 289G - 460H, \\
\zeta(\bar{1}, 3, \bar{1}) &= -11820A + 1614B - 1120C + 23726D - 16460E - 557F - 27466G + 43926H, \\
\zeta(\bar{1}, \bar{3}, 1) &= 6380A - 874B + 612C - 12776D + 8880E + 302F + 14858G - 23738H, \\
\zeta(\bar{1}, \bar{3}, \bar{1}) &= 12610A - 1722B + 1194C - 25312D + 17560E + 594F + 29302G - 46862H, \\
\zeta(\bar{1}, 2, 2) &= -190A + 26B - 18C + 384D - 266E - 9F - 442G + 708H, \\
\zeta(\bar{1}, 2, \bar{2}) &= 13726A - 1880B + 1314C - 27494D + 19106E + 649F + 31960G - 51066H, \\
\zeta(\bar{1}, \bar{2}, 2) &= -13631A + 1867B - 1305C + 27302D - 18973E - 644F - 31739G + 50712H, \\
\zeta(\bar{1}, \bar{2}, \bar{2}) &= 599A - 82B + 57C - 1203D + 835E + 28F + 1394G - 2229H, \\
\zeta(\bar{1}, 1, 3) &= -3186A + 435B - 300C + 6399D - 4438E - 150F - 7401G + 11839H, \\
\zeta(\bar{1}, 1, \bar{3}) &= -2732A + 376B - 268C + 5448D - 3798E - 130F - 6384G + 10182H, \\
\zeta(\bar{1}, \bar{1}, 3) &= 20431A - 2799B + 1969C - 40888D + 28427E + 966F + 47591G - 76018H, \\
\zeta(\bar{1}, \bar{1}, \bar{3}) &= -7808A + 1070B - 758C + 15608D - 10858E - 369F - 18196G + 29054H.
\end{aligned}$$

*For length four,*

$$\begin{aligned}
 \zeta(2, 1, 1, 1) &= -13504A + 1856B - 1344C + 26880D - 18752E - 640F - 31552G + 50304H, \\
 \zeta(2, 1, 1, \bar{1}) &= -11109A + 1518B - 1044C + 22320D - 15477E - 523F - 25812G + 41289H, \\
 \zeta(2, 1, \bar{1}, 1) &= 1174A - 158B + 101C - 2395D + 1642E + 54F + 2691G - 4333H, \\
 \zeta(2, 1, \bar{1}, \bar{1}) &= 14927A - 2044B + 1431C - 29899D + 20773E + 705F + 34745G - 55518H, \\
 \zeta(2, \bar{1}, 1, 1) &= -2712A + 371B - 258C + 5439D - 3777E - 128F - 6306G + 10083H, \\
 \zeta(2, \bar{1}, 1, \bar{1}) &= -14828A + 2030B - 1419C + 29709D - 20641E - 701F - 34517G + 55158H, \\
 \zeta(2, \bar{1}, \bar{1}, 1) &= -7120A + 977B - 681C + 14262D - 9911E - 337F - 16585G + 26496H, \\
 \zeta(2, \bar{1}, \bar{1}, \bar{1}) &= 11204A - 1534B + 1074C - 22440D + 15595E + 530F + 26096G - 41691H, \\
 \zeta(\bar{2}, \bar{1}, 1, 1) &= 8717A - 1197B + 847C - 17415D + 12122E + 412F + 20334G - 32456H, \\
 \zeta(\bar{2}, \bar{1}, 1, \bar{1}) &= -8511A + 1162B - 806C + 17085D - 11852E - 401F - 19775G + 31627H, \\
 \zeta(\bar{2}, \bar{1}, \bar{1}, 1) &= 3432A - 470B + 327C - 6882D + 4779E + 162F + 7980G - 12759H, \\
 \zeta(\bar{2}, \bar{1}, \bar{1}, \bar{1}) &= -652A + 89B - 66C + 1296D - 905E - 31F - 1531G + 2436H, \\
 \zeta(\bar{2}, 1, \bar{1}, 1) &= 8659A - 1183B + 822C - 17376D + 12059E + 409F + 20134G - 32193H, \\
 \zeta(\bar{2}, 1, 1, \bar{1}) &= 3571A - 490B + 344C - 7145D + 4969E + 169F + 8322G - 13291H, \\
 \zeta(\bar{1}, 2, 1, 1) &= 190A - 26B + 18C - 384D + 265E + 9F + 442G - 707H, \\
 \zeta(\bar{1}, 2, 1, \bar{1}) &= 190A - 25B + 18C - 385D + 265E + 9F + 442G - 707H, \\
 \zeta(\bar{1}, 2, \bar{1}, 1) &= -27776A + 3801B - 2654C + 55667D - 38668E - 1313F - 64648G + 103316H, \\
 \zeta(\bar{1}, 2, \bar{1}, \bar{1}) &= -19006A + 2604B - 1828C + 38048D - 26452E - 900F - 44288G + 70740H, \\
 \zeta(\bar{1}, \bar{2}, 1, 1) &= -2407A + 330B - 233C + 4812D - 3347E - 113F - 5610G + 8957H, \\
 \zeta(\bar{1}, \bar{2}, 1, \bar{1}) &= -202A + 32B - 34C + 353D - 274E - 12F - 533G + 807H, \\
 \zeta(\bar{1}, \bar{2}, \bar{1}, 1) &= 6507A - 893B + 631C - 13009D + 9054E + 309F + 15184G - 24238H, \\
 \zeta(\bar{1}, \bar{2}, \bar{1}, \bar{1}) &= 31628A - 4333B + 3038C - 63328D + 44021E + 1497F + 73681G - 117702H, \\
 \zeta(\bar{1}, 1, 2, 1) &= -122A + 17B - 13C + 242D - 170E - 6F - 288G + 458H, \\
 \zeta(\bar{1}, 1, 2, \bar{1}) &= 3310A - 453B + 314C - 6641D + 4609E + 156F + 7692G - 12301H, \\
 \zeta(\bar{1}, 1, \bar{2}, \bar{1}) &= 6195A - 850B + 600C - 12383D + 8619E + 294F + 14454G - 23073H, \\
 \zeta(\bar{1}, 1, \bar{2}, 1) &= 2888A - 394B + 272C - 5803D + 4023E + 136F + 6706G - 10729H, \\
 \zeta(\bar{1}, \bar{1}, 2, 1) &= -7433A + 1019B - 711C + 14888D - 10348E - 352F - 17315G + 27663H, \\
 \zeta(\bar{1}, \bar{1}, 2, \bar{1}) &= 6793A - 932B + 658C - 13586D + 9453E + 322F + 15848G - 25301H, \\
 \zeta(\bar{1}, \bar{1}, \bar{2}, 1) &= -313A + 43B - 30C + 626D - 437E - 15F - 730G + 1167H, \\
 \zeta(\bar{1}, \bar{1}, \bar{2}, \bar{1}) &= -18914A + 2592B - 1822C + 37855D - 26321E - 895F - 44073G + 70394H, \\
 \zeta(\bar{1}, 1, 1, 2) &= 191A - 26B + 18C - 384D + 267E + 9F + 442G - 709H, \\
 \zeta(\bar{1}, 1, 1, \bar{2}) &= -2521A + 345B - 240C + 5054D - 3510E - 119F - 5864G + 9374H, \\
 \zeta(\bar{1}, 1, \bar{1}, 2) &= 13126A - 1798B + 1257C - 26291D + 18271E + 621F + 30567G - 48838H, \\
 \zeta(\bar{1}, \bar{1}, 1, 2) &= 13312A - 1826B + 1295C - 26595D + 18512E + 630F + 31043G - 49555H, \\
 \zeta(\bar{1}, \bar{1}, 1, \bar{2}) &= -4812A + 661B - 475C + 9593D - 6687E - 228F - 11237G + 17924H, \\
 \zeta(\bar{1}, \bar{1}, \bar{1}, \bar{2}) &= -13127A + 1798B - 1258C + 26291D - 18271E - 621F - 30565G + 48836H.
 \end{aligned}$$

For length five,

$$\begin{aligned}
\zeta(\bar{1}, 1, 1, 1, 1) &= -191A + 26B - 18C - 442G + 384D - 266E - 9F + 709H, \\
\zeta(\bar{1}, 1, 1, 1, \bar{1}) &= -191A + 26B - 18C + 385D - 266E - 9F - 442G + 709H, \\
\zeta(\bar{1}, 1, 1, \bar{1}, 1) &= 4481A - 614B + 430C - 8973D + 6237E + 212F + 10438G - 16674H, \\
\zeta(\bar{1}, 1, 1, \bar{1}, \bar{1}) &= -A - E + 2H, \\
\zeta(\bar{1}, 1, \bar{1}, 1, 1) &= -4693A + 643B - 451C + 9395D - 6531E - 222F - 10930G + 17462H, \\
\zeta(\bar{1}, 1, \bar{1}, \bar{1}, 1) &= -313A + 43B - 31C - 730G + 626D - 436E - 15F + 1167H, \\
\zeta(\bar{1}, 1, \bar{1}, \bar{1}, \bar{1}) &= -13126A + 1798B - 1258C + 26291D - 18271E - 621F - 30565G + 48837H, \\
\zeta(\bar{1}, \bar{1}, 1, 1, 1) &= 7496A - 1031B + 747C - 14915D + 10408E + 355F + 17522G - 27929H, \\
\zeta(\bar{1}, \bar{1}, 1, 1, \bar{1}) &= 2081A - 285B + 194C - 4183D + 2901E + 98F + 4840G - 7740H, \\
\zeta(\bar{1}, \bar{1}, 1, \bar{1}, 1) &= -3308A + 452B - 313C + 6641D - 4607E - 156F - 7689G + 12297H, \\
\zeta(\bar{1}, \bar{1}, 1, \bar{1}, \bar{1}) &= -12121A + 1660B - 1164C + 24269D - 16868E - 573F - 28225G + 45094H, \\
\zeta(\bar{1}, \bar{1}, \bar{1}, 1, 1) &= 12622A - 1729B + 1210C - 25280D + 17569E + 597F + 29393G - 46961H, \\
\zeta(\bar{1}, \bar{1}, \bar{1}, \bar{1}, 1) &= 10552A - 1445B + 1008C - 21144D + 14690E + 499F + 24565G - 39254H.
\end{aligned}$$

#### 4. WEIGHT ONE

The relations in weight one are crucial for higher level cases because they provide the seeded relations. To simplify notation we use  $L_N(j)$  to denote  $L_N(1|j)$  only in this section. There are  $N - 1$  MPVs of weight 1 and level  $N$ :

$$L_N(j), \quad 0 < j < N.$$

Taking  $\mathbb{C} \setminus (-\infty, 0]$  as the domain of the single valued logarithm, namely,  $\log(z) = \log|z| + i\text{Arg}(z)$  where  $-\pi < \text{Arg}(z) < \pi$ , then we have

$$L_N(j) = Li_1(\mu^j) = -\log(1 - \mu^j).$$

There are many linear relations among these numbers. For instance, if  $j < N/2$  then we have the symmetric relation

$$-\log(1 - \mu^j) = -\log(1 - \mu^{N-j}) - \log(-\mu^j) = -\log(1 - \mu^{N-j}) + \frac{N-2j}{N}\pi i.$$

Thus for all  $1 < j < N/2$

$$(4) \quad (N-2)(L_N(j) - L_N(N-j)) = (N-2j)(L_N(1) - L_N(N-1)).$$

Further, from [1, (B)] for any divisor  $k$  of  $N$  and  $1 \leq a < k$  we have the distribution relation

$$(5) \quad \sum_{0 \leq j < N/k} L_N(a + kj) = L_N(aN/k).$$

It follows from the main result of Bass [1] corrected by Ennola [8] that all the linear relations between  $L_N(j)$  are consequences of (4) and (5). Moreover, the  $\mathbb{Q}$ -dimension of  $\langle L_N(j), 0 < j < N \rangle$  is  $d(1, N) = \phi(N)/2 + \nu(N) - 1$  where  $\nu(N)$  is the number of prime divisors of  $N$ . Computation by (4) and (5) shows that our Main Problem has affirmative answers if the weight  $w = 1$  and the level  $N < 102$ . We also randomly checked the cases where  $N = 500, 501$  and  $N = 1000$  by MAPLE. For example:

$$\begin{aligned}
 L_8(1) &= L_8(2) - L_8(5), \\
 L_8(4) &= 2L_8(2) - 2L_8(3) + 2L_8(5), \\
 L_8(7) &= L_8(2) - 3L_8(3) + 2L_8(5), \\
 L_8(6) &= L_8(2) - 2L_8(3) + 2L_8(5).
 \end{aligned}$$

To deal with the general case, we define

$$\Phi_N := \{j \in \mathbb{Z} : 1 \leq j \leq \lfloor N/2 \rfloor + 1, \gcd(j, N) = 1\}.$$

**Theorem 4.1.** *If  $N = p^n$  for some prime  $p$  then the set  $\mathbf{B} := \{L_N(j) : j \in \Phi_N\}$  is a  $\mathbb{Z}$ -basis of  $\mathcal{MPV}_{\mathbb{Z}}(1, N)$ .*

*Proof.* First we assume  $p$  is an odd prime. Taking  $j = (N - 1)/2$  in (4) we get:

$$L_N(1) - L_N(N - 1) = (N - 2) \left( L_N((N - 1)/2) - L_N((N + 1)/2) \right).$$

So  $L_N(N - 1)$  is a  $\mathbb{Z}$ -linear combination of elements in  $\mathbf{B}$ . Now for any  $j > (N + 1)/2$  and  $(j, p) = 1$  equation (4) yields:

$$\begin{aligned}
 (N - 2)(L_N(j) - L_N(N - j)) &= (N - 2j)L_N(1) - L_N(N - 1) \\
 &= (N - 2j)(N - 2) \left( L_N((N - 1)/2) - L_N((N + 1)/2) \right).
 \end{aligned}$$

Thus

$$L_N(j) = L_N(N - j) + (N - 2j) \left( L_N((N - 1)/2) - L_N((N + 1)/2) \right).$$

This shows that for every  $j < N$  and  $(j, p) = 1$  the value  $L_N(j)$  is a  $\mathbb{Z}$ -linear combination of  $\mathbf{B}$ . Now if  $j = ap^t$  with  $(a, p) = 1$  then from (5) we see immediately that  $L_N(j)$  is a linearly combination of  $\mathbf{B}$ , too. This finishes the proof of part (a) the theorem since clearly  $|\mathbf{B}| = \varphi(N)/2 + 1$  which is equal to  $\dim \mathcal{MPV}(1, N)$ .

When  $p = 2$  the proof is completely similar to the above. We leave it to the interested reader.  $\square$

If  $N$  has at least two distinct prime factor then the situation becomes much more involved. First we need an easy result from elementary number theory.

**Theorem 4.2.** *(a) Let  $p < q$  be two distinct odd primes and  $N = pq$ . Denote by  $Q(p, q)$  the quotient group  $\mathbb{Z}_p^\times / \langle -1, q \rangle$ . Then the set*

$$\begin{aligned}
 \mathbf{B} := & \left( \left( \{L_N(j) : j \in \Phi_N\} \cup \{L_N(ap) : \tilde{a} \in Q(q, p)\} \right) \setminus \{L_N(a) : \tilde{a} \in Q(q, p)\} \right) \\
 & \cup \{L_N(bq) : \tilde{b} \in Q(p, q)\} \setminus \{L_N(b) : \tilde{b} \in Q(p, q)\}
 \end{aligned}$$

*is a  $\mathbb{Z}$ -basis of  $\mathcal{MPV}(1, N)$ . Here each  $a$  is chosen to be a fixed representatives in  $[1, q-1]$  for  $\tilde{a} \in Q(q, p)$  and **then** each  $b$  is chosen to be as a representative in  $[1, q+p-1] \setminus \{q-1\} \setminus \{a > 1 : \tilde{a} \in Q(q, p)\}$  for  $\tilde{b} \in Q(p, q)$ . Further, we always require that  $p$  represents*

$\langle -1, p \rangle \in Q(q, p)$  and  $q$  represents  $\langle -1, q \rangle \in Q(p, q)$ . Hence  $L_N(p), L_N(q) \in \mathbf{B}$  while  $L_N(1) \notin \mathbf{B}$ .

*Remark 4.3.* Roughly speaking, since we need both  $L_N(p)$  and  $L_N(q)$  but only took away  $L_N(1)$  in step (iii) we have to take away one more element to balance out. This element has to be  $L_N(\lfloor N/2 \rfloor + 1)$ .

*Proof.* We carry out the proof in several steps. We will start from  $S_1 = \{j : 1 \leq j < N\}$  and construct smaller and smaller subsets  $S_1 \supset S_2 \supset \cdots \supset S_\ell$  such that  $L_N(j) \in \mathbb{Z}[L_N(k) : k \in S_{\alpha+1}]$  for all  $j \in S_\alpha \setminus S_{\alpha+1}$ , and  $\mathbf{B} = \{L_N(j) : j \in S_\ell\}$ .

(i). Let  $S_2 = \{j \in S_1 : \text{if a prime square } \mathfrak{p}^2 | j \text{ then } \mathfrak{p} \neq p, q\}$ . If  $j = ap^e$  with  $e > 1$  and  $\gcd(a, p) = 1$  then by (5) (take  $a$  to be  $ap^{e-1}$  there)

$$L_N(ap^e) = \sum_{0 \leq l \leq p-1} L_N(ap^{e-1} + lq).$$

Clearly the highest power of  $p$  dividing some  $ap^{e-1} + lq$  is when  $l = 0$  while all the others are prime to  $p$ . Using (5) repeatedly and doing the same for  $q$  we see that we can  $S_2$  as above.

(ii). First we consider the following ‘‘cyclic pattern’’ in  $S_2$ . Let  $a \in [1, q-1]$ ,  $\gcd(a, p) = 1$ . Then by (5) we have

$$(6) \quad L_N(ap) = \sum_{0 \leq l \leq p-1} L_N(a + lq) = L_N(a) + \sum_{1 \leq l \leq p-1} L_N(a + lq).$$

Clearly  $\gcd(q, a + lq) = 1$ . But it is possible that for some  $l_0$  we have  $p | (a + l_0q)$ , namely

$$a + l_0q \equiv 0 \pmod{p}.$$

In fact, there is a unique  $l_0 \in [1, p-1]$  such that this is true. Take this  $l_0$  and write  $a_1p^{e_1} = a + l_0q$ ,  $a_1 \in [1, q-1]$ ,  $\gcd(a_1, p) = 1$ . Repeating this we will produce a sequence of numbers  $a, a_1, a_2, \dots$  all lying in the finite set  $[1, q-1]$ . So there is a smallest positive integer  $s$  such that  $a = a_s$ . The relations between these numbers are given by

$$a \equiv a_1p, \quad a_1 \equiv a_2p, \quad \dots, \quad a_{s-1} \equiv ap \pmod{q}.$$

Hence  $s$  is the smallest positive integer satisfying

$$a \equiv ap^s \pmod{q}$$

Since  $1 \leq a < q$  we know that  $s$  is the order of the multiplicative group  $\langle p \rangle$  viewed as a cyclic subgroup of  $\mathbb{Z}_q^\times$ . Therefore we only need to pick one representative  $ap$  from  $S_2$  for the class  $a\langle p \rangle \in \mathbb{Z}_q^\times / \langle p \rangle$ . Moreover, the above argument shows that

$$L_N(ap) = L(a) + L(a_1p) + \sum_{j \in J} L_N(j)$$

by (i), where  $J \subset S_2$  such that  $\gcd(j, N) = 1$  and  $N > j \geq a + q > a$  for all  $j \in J$ . Repeating this we get

$$(7) \quad L_N(ap) = L_N(a) + L(a_sp) + \sum_{j \in J} L_N(j) + \sum_{m=1}^{s-1} \sum_{j \in J_m} L_N(j)$$

where  $J_m \subset S_2$  such that  $\gcd(j, N) = 1$  and  $N > j \equiv a_m \not\equiv a \pmod{q}$  for all  $j \in J_m$  and all  $m = 1, \dots, s-1$ . Hence  $L_N(a)$  appears in (7) only once while  $L(ap)$  and  $L(a_s p)$  cancel. Further, for the class  $(q-1)\langle p \rangle$  we can choose the representative  $a < q-1$ . Otherwise  $q-1$  must be the only representative which means  $q-1 \equiv (q-1)p \equiv q-p \pmod{q}$ . This is absurd since we assumed  $p < q$ . So we can take the following subset of  $S_2$ :

$$S_3 = \left( \{j \in \mathbb{Z} : 1 \leq j < N, \gcd(j, N) = 1\} \cup \{bq : 1 \leq b \leq p-1\} \right. \\ \left. \cup \{ap : \tilde{a} \in Q'(q, p)\} \right) \setminus \{a : \tilde{a} \in Q'(q, p)\},$$

where  $Q'(p, q)$  is the quotient group  $\mathbb{Z}_p^\times / \langle q \rangle$ .

(iii). We can do almost exactly the same for  $q$  as we did in (ii) except that if we take representatives  $b$  in just  $[1, p-1]$  we may run into trouble because such a number may be already taken out in (ii). So we change our range for the choices of  $b$ 's to  $[1, q+p-2] \setminus \{a > 1 : \tilde{a} \in Q'(q, p)\}$  to guarantee this won't happen. We further take away  $q-1$  for the sake of the next step. Thus we can now take

$$S_4 := \left( \left( (j : 1 \leq j < N, \gcd(j, N) = 1) \cup \{ap : \tilde{a} \in Q'(q, p)\} \right) \setminus \{a : \tilde{a} \in Q'(q, p)\} \right) \\ \left. \cup \{bq : \tilde{b} \in Q'(p, q)\} \right) \setminus \{b : \tilde{b} \in Q'(p, q)\}$$

(iv). We now need to use the symmetry (4) to cut the size of  $S_4$  further. In (4) taking  $j = (N-1)/2$  we get:

$$L_N(1) - L_N(N-1) = (N-2) \left( L_N((N-1)/2) - L_N((N+1)/2) \right).$$

Now for any  $j > (N+1)/2$  equation (4) yields:

$$L_N(j) = L_N(N-j) + \frac{N-2j}{N-2} (L_N(1) - L_N(N-1)) \\ (8) \quad = L_N(N-j) + (N-2j) \left( L_N((N-1)/2) - L_N((N+1)/2) \right).$$

Applying this to all  $j$  of the form  $j = ap, bq \in S_4$  we can reduce  $S_4$  to

$$S_5 := \left( \left( (j : 1 \leq j < N, \gcd(j, N) = 1) \cup \{ap : \tilde{a} \in Q(q, p)\} \right) \setminus \{a : \tilde{a} \in Q'(q, p)\} \right) \\ \left. \cup \{bq : \tilde{b} \in Q(p, q)\} \right) \setminus \{b : \tilde{b} \in Q'(p, q)\}$$

(v). Note that if  $a \notin S_5$  we have to deal with  $N-a$  every carefully. We can not use (8) because it only implies that  $L_N(N-a)$  is a  $\mathbb{Z}$ -linear combination of  $L_N(a)$ ,  $L_N((N-1)/2)$  and  $L_N((N+1)/2)$ . But  $L_N(a) \notin S_5$  and we don't know apriori whether  $L_N(a)$  is of the form  $L_N(N-a)$  plus other elements in  $S_5$  (which would cause  $L_N(N-a)$  to cancel).

Replacing  $a$  by  $a' = q - a$  in (6) we get

$$L_N(a'p) = L_N(a') + \sum_{1 \leq l \leq p-2} L_N(a' + lq) + L_N(N - a).$$

By the same process to derive (7) we get an similar equation

$$(9) \quad L_N(a'p) = L_N(a') + L(a'_s p) + \sum_{m=0}^{s-1} \sum_{j \in J_m} L_N(j) + L_N(N - a),$$

where each  $j \in J_0$  satisfies  $N - a > j \geq a' + q > q + p - 1$  (since  $p < q$ ) and  $(j, N) = 1$ , and for  $m = 1, \dots, s-1$  each  $j \in J_m$  satisfies  $j \equiv a'_m \pmod{q}$  and  $(j, N) = 1$ . We want to show  $L_N(N - a)$  only appears once in (9).

Since all  $a'_m < q - 1$ ,  $N - a > j \geq a' + q > q + 1$ . Thus if  $j \in J_0$  then  $j \in S_4$  (not removed in step (iii) above). If  $j \in J_m$  then it cannot be removed when picking  $ap$ 's. However, it could be removed when picking  $bq$ 's. Suppose this does happen. Then we need to show that  $L_N(N - a)$  does not appear in the  $\mathbb{Z}$ -linear combination expression of  $L_N(j)$ . Suppose on the contrary we have the express

$$L_N(jq) = L_N(j) + L(j_t q) + \sum_{n=0}^{t-1} \sum_{k \in I_n} L_N(k),$$

where  $k = N - a$  for some  $k$ ,  $k \equiv j_n \pmod{p}$  for all  $k \in I_n$ ,  $\gcd(k, N) = 1$ ,  $j \equiv j_1 q$ ,  $j_1 \equiv j_2 q$ ,  $j_2 \equiv j_3 q \pmod{p}$ , and  $q^t \equiv 1 \pmod{p}$ .

Again  $L_N(a'p)$  and  $L(a'_s p)$  cancel. Since  $a' \not\equiv q - a \pmod{q}$  and in particular  $q - a$  is a different representative of  $a\langle -1, p \rangle$  which yields  $q - a \in S_4$ . We see immediately that  $L_N(N - a)$  is a  $\mathbb{Z}$ -linear combination of the other terms in  $S_4$ , namely we can take  $S_5 = S_4 \setminus \{N - a\}$ .

By considering the quotient groups  $Q(q, p)$  and  $Q(p, q)$  we see that if  $a < q - 1$  is the chosen representative of  $a\langle -1, p \rangle$  then  $a \not\equiv q - a \pmod{q}$  and in particular  $q - a$  is a different representative of  $a\langle -1, p \rangle$ . Similarly if  $b < q + p - 1$  then  $q - b$  is a different representative of  $b\langle -1, q \rangle$ . We can mimic step (iv) to remove  $N - a$  and  $N - b$ .  $\square$

**Theorem 4.4.** *Let  $N$  be a positive integer. Then there is a  $\mathbb{Z}$ -basis of  $\mathcal{MPV}_{\mathbb{Z}}(1, N)$ .*

*Proof.* The main idea is to use [5, Thm. 4.6] together with the following fact. If  $N$  is odd then taking  $j = (N + 1)/2$  we get

$$\frac{1}{N} \pi \sqrt{-1} = \left( \frac{2j}{N} - 1 \right) \pi \sqrt{-1} = \log(-\mu^j) = \log(1 - \mu^j) - \log(1 - \mu^{-j}).$$

If  $N$  is even then taking  $j = (N + 2)/2$  we get

$$\frac{2}{N} \pi \sqrt{-1} = \left( \frac{2j}{N} - 1 \right) \pi \sqrt{-1} = \log(-\mu^j) = \log(1 - \mu^j) - \log(1 - \mu^{-j}).$$

Thus for any  $k$  we see that  $\log(-\mu^k)$  is an integral multiple of  $\frac{1}{N} \pi \sqrt{-1}$  (resp.  $\frac{2}{N} \pi \sqrt{-1}$ ) if  $N$  is odd (resp. even).  $\square$

## 5. WEIGHT TWO

In [11] we studied the dimension  $d(2, N)$  using the following definition.

**Definition 5.1.** We call the level  $N$  *standard* if either (i)  $N = 1, 2$  or  $3$ , or (ii)  $N$  is a prime power  $p^n$  ( $p \geq 5$ ). Otherwise  $N$  is called *non-standard*.

We believe the following statement is true ([11, Conj. 10.3]).

**Conjecture 5.2.** *If  $N$  is a standard level then the standard relations always provide the sharp bounds of  $d(w, N)$ , namely, all linear relations can be derived from the standard ones. If  $N$  is a non-standard level then the bound in [7, Cor. 5.25] is sharp and the non-standard relations exist in  $\mathcal{MPV}(w, N)$  for all  $w \geq 2$  and sufficiently large  $N$  (depending on  $w$ ).*

Because we don't know what explicit relations to use to obtain the sharp bound in non-standard levels we have to restrict ourselves to the standard ones in which case we have the following result.

**Theorem 5.3.** ([10, Thm. 1]) *Let  $p \geq 5$  be a prime. Then  $d(2, p) \leq (5p + 7)(p + 1)/24$ . If Grothendieck's period conjecture ([7, 5.27(c)]) is true then the equality holds and the standard relations in  $\mathcal{MPV}(2, p)$  imply all the others.*

Assuming Grothendieck's period conjecture is true we now can show the existence of  $\mathbb{Z}$ -basis in  $\mathcal{MPV}(2, N)$  for  $N$  up to 13. In fact, we can do a little more.

Let  $p$  be a prime  $p \geq 5$ . Let  $h = (p - 1)/2$ . Let  $\mathbf{B}$  be some set of MPVs so that the  $\mathbb{Z}$ -span  $\mathbb{Z}[\mathbf{B}]$  gives the  $\mathbb{Z}$ -module  $\mathcal{MPV}_{\mathbb{Z}}(2, p)$ . Let's start with  $\mathbf{B}$  equal to the set of all MPVs of weight 2 and level  $p$ . Ideally we want to narrow  $\mathbf{B}$  down to have only  $(5p + 7)(p + 1)/24$  elements.

We have by RDS [11, (8.2)]

$$(10) \quad L_N(1, 1|i, 0) + L_N(2|i) = L_N(1, 1|i, -i) \quad \forall 1 \leq i \leq p - 1.$$

So we can express the dilog  $L_N(2|i)$  as a  $\mathbb{Z}$ -linear combination of double log values, for  $1 \leq i \leq p - 1$ . We also have by FDS [11, (8.1)]:  $\forall 1 \leq i, j \leq p - 1$

$$(11) \quad L_N(2|i + j) + L_N(1, 1|i, j) + L_N(1, 1|j, i) = L_N(1, 1|i, j - i) + L_N(1, 1|j, i - j).$$

Taking  $i = 1, j = p - 1$ , say, we can remove  $L_N(2|0)$  from  $\mathbf{B}$ , too. Moreover, combining (10) and (11) we get  $\forall 1 \leq i, j \leq p - 1, i + j \neq 0$

$$(12) \quad \begin{aligned} L_N(1, 1|i + j, -i - j) + L_N(1, 1|i, j) + L_N(1, 1|j, i) \\ = L_N(1, 1|i + j, 0) + L_N(1, 1|i, j - i) + L_N(1, 1|j, i - j). \end{aligned}$$

Now weight one relations yield [11, (8.7)]:  $\forall 1 \leq i, j \leq p - 1$

$$(13) \quad \begin{aligned} L_N(1, 1|i, j - i) + L_N(1, 1|j, i - j) - L_N(1, 1|i, -j - i) - L_N(1, 1|-j, i + j) \\ = (p - 2j)(L_N(1, 1|i, h - i) + L_N(1, 1|h, i - h) - L_N(1, 1|i, -i - h) - L_N(1, 1|-h, i + h)). \end{aligned}$$

In the following, to save space, I'll write  $(i, j)$  for  $L_N(1, 1|i, j)$ . Rewrite (12) and (13) as:  $\forall 1 \leq i, j \leq p-1, i+j \neq 0$

$$(14) \quad (i+j, -i-j) + (i, j) + (j, i) = (i+j, 0) + (i, j-i) + (j, i-j),$$

and  $\forall 1 \leq i, j \leq p-1$

$$(15) \quad (i, j-i) + (j, i-j) - (i, -j-i) - (-j, i+j) \\ = (p-2j)((i, h-i) + (h, i-h) - (i, -i-h) - (h+1, i+h)).$$

Define the subsets:

$$\begin{aligned} S_1 &:= \{(a, 0) : 1 \leq a \leq p-1\}, \\ S_2 &:= \{(a, p-a) : 1 \leq a \leq p-1\}, \\ S_3 &:= \{(h, b) : 1 \leq b \leq p-1, b \neq h+1\}, \\ S_4 &:= \{(h+1, b) : 1 \leq b \leq p-1, b \neq h\}, \end{aligned}$$

Taking  $i = h$  in (15) we get

$$(16) \quad (h, j-h) + (j, h-j) - (h, -j-h) - (-j, h+j) \\ = (p-2j)(2(h, 0) - (h, 1) - (h+1, -1)).$$

So we may define

$$S_5 := \{(a, h-a) : 1 \leq a \leq h-1\}.$$

Taking  $i = h+1$  in (15) we get

$$(17) \quad (h+1, h+j) + (j, h+1-j) - (h+1, h-j) - (-j, h+1+j) \\ = (p-2j)((h+1, -1) + (h, 1) - 2(h+1, 0)).$$

So we may define

$$S_6 := \{(a, h+1-a) : 1 \leq a \leq h-1\}.$$

Set

$$S = \bigcup_{j=1}^6 S_j.$$

Note  $|S|$  is about  $5p$ . Denote by  $x \sim y$  if  $x - y \in \mathbb{Z}[S]$ . Then (14) and (15) become:

$$(18) \quad (i, j) + (j, i) \sim (i, j-i) + (j, i-j) \quad \forall 1 \leq i, j \leq p-1, i+j \neq p,$$

and

$$(19) \quad (i, j-i) + (j, i-j) \sim (i, -j-i) + (-j, i+j) \quad \forall 1 \leq i, j \leq p-1.$$

Think  $(i, j)$  as lattice points on the torus  $(\mathbb{R}/p\mathbb{Z})^2$ . On  $xy$ -plane the two relations (18) and (19) should shrink the size of  $\mathbf{B}$  to roughly  $p^2/4$ . But we can't get exactly this yet. Combining (18) and (19) we have

$$(20) \quad (i, j) + (j, i) \sim (i, -j-i) + (-j, i+j) \quad \forall 1 \leq i, j \leq p-1, i+j \neq p.$$

Change  $j$  to  $-j$  in (20) we have

$$(21) \quad (i, -j) + (-j, i) \sim (j, i - j) + (i, j - i) \quad \forall 1 \leq i, j \leq p - 1, i - j \neq p.$$

Comparing (18) to (21) we get

$$(22) \quad (i, j) + (j, i) \sim (i, -j) + (-j, i) \quad \forall 1 \leq i, j \leq p - 1, i \pm j \neq p,$$

Setting  $i = a + b, j = -a$  in (20) we have

$$(23) \quad (a, b) \sim (a + b, -a) + (-a, a + b) - (a + b, -b).$$

Define  $X = \{(a, b) : a, b > 0, a + b < p\}$  and  $Y = \{(a, b) : 1 \leq a, b < p, a + b > p\}$ . If  $(a, b) \in Y$  then

$$(24) \quad (a, b) \sim (a + b - p, p - a) + (p - a, a + b - p) - (a + b - p, p - b) \in X.$$

This shrinks the size  $\mathbf{B}$  to about  $p^2/2$ . Further, if  $(i, j) \in X$  and  $j > p/2$  then by (22)

$$(25) \quad (i, j) \sim (i, p - j) + (p - j, i) - (j, i) \in X.$$

By this relation we can remove the top 1/4 of the triangle of  $X$ . This further cuts the size of  $\mathbf{B}$  to  $3p^2/8$ , still  $p^2/6$  away from the ideal size.

Applying (23) to each of the terms in (22) we get

$$(26) \quad (-i, i + j) + (-j, i + j) \sim (-i, i - j) + (j, i - j).$$

By substitution  $(a, b) = (-j, i + j)$  in (26) we get

$$(27) \quad (a, b) \sim (-a - b, a + 2b) + (-a, a + 2b) - (-a - b, b)$$

Define

$$(28) \quad R_1 = \{(x, y) \in X : x + 4y < 2p, x + 2y \geq p, x > y\},$$

$$(29) \quad R_2 = \{(x, y) \in X : x + 4y < 2p, x + 2y \geq p, 3y - x > p\} \subset \{x < y\},$$

$$(30) \quad L_0 = \{(x, y) \in X : x + 4y > 2p\},$$

$$(31) \quad L_1 = \{(x, y) \in X : x + 4y < 2p, x + 2y \geq p, 3y - x < p, x < y\} \subset \{x < y\},$$

$$(32) \quad L_2 = \{(x, y) \in X : x < y, x + 2y < p\} \subset \{x < y\},$$

$$(33) \quad L_3 = \{(x, y) \in X : x < y, x + 4y > 2p\} \subset \{x < y\}.$$

If  $(a, b) \in R_1$  we get from (27)

$$(34) \quad (a, b) \sim (p - a - b, a + 2b - p) + (p - a, a + 2b - p) - (p - a - b, b) \in L_2.$$

If  $(a, b) \in R_2$  we get from (27)

$$(35) \quad (a, b) \sim (p - a - b, a + 2b - p) + (p - a, a + 2b - p) - (p - a - b, b) \in L_2 \cup L_3.$$

Note if  $(a, b) \in X$  but  $2a + b < p$  then we can't use (27) directly since the first two elements on the right are not in  $X$  anymore. But we can use (24) to transform them to

$X$ :

$$(p - a - b, a + 2b) \sim (b, a + b) + (a + b, b) - (b, p - a - 2b) = (x_1, y_1) + (x_2, y_2) + (x_3, y_3) \in X,$$

$$(p - a, a + 2b) \sim (2b, a) + (a, 2b) - (2b, p - a - 2b) = (x_4, y_4) + (x_5, y_5) + (x_6, y_6) \in X.$$

Let  $(x_7, y_7) = (p - a - b, b)$ . Now define

$$R_3 = \{(x, y) \in X : 2x + 4y > p, 2x + 2y < p, x + 4y < px > y\},$$

$$L_4 = \{(x, y) \in X : x + 2y < p, 2x + 2y > p, x > y\}.$$

Note  $L_4$  is between  $R_1$  and  $R_3$  on  $\{x > y\} \cap X$ . Then for all  $(a, b) \in R_3$  we have

$$(a, b) \sim \sum_{i=1}^7 (x_i, y_i).$$

We can check easily that  $(x_1, y_1) \in \{x < y\} \cap X$ . If it lies in  $R_2$  then (35) transforms it to regions in  $L_2 \cup L_3$ . Similarly if  $(x_i, y_i) \in L_4$  then we're fine. So we now assume  $x_i > y_i$  when we consider  $(x_i, y_i)$ . Then we find that  $(x_i, y_i) \in L_4$  for  $i = 2, 4, 5, 7$ . Indeed,  $(a, b) \in R_3$  implies that

$$2a + 4b > p, \quad 2a + 2b < p, \quad a > b.$$

Then

$$x_2 + 2y_2 = a + 3b < 2a + 2b < p,$$

$$2x_2 + 2y_2 = 4b + 2a > p,$$

$$x_3 + 2y_3 = 2p - 2a - 3b < 2p - a - 4b < p \quad (x_3 > y_3),$$

$$2x_3 + 2y_3 = 2p - 2a - 2b > p,$$

$$x_4 + 2y_4 = 2a + 2b < p,$$

$$2x_4 + 2y_4 = 4b + 2a > p,$$

$$x_5 + 2y_5 = 2a + 2b - (a - 2b) < p \quad (x_5 > y_5),$$

$$2x_5 + 2y_5 = 2a + 4b > p,$$

$$x_7 + 2y_7 = p - a + b < p,$$

$$2x_7 + 2y_7 = 2p - 2a > p + (p - 2a - 2b) > p.$$

Now for the remaining element  $(x_6, y_6)$ :

$$2x_6 + 2y_6 = 2p - 2a > p.$$

But  $x_6 + 2y_6 = 2p - 2a - 2b > p$ . This means that if we require  $x_6 > y_6$  then  $a + 4b > p$  which implies that  $2a + 3b > a + 4b > p$ , and thus  $x_6 + 4y_6 = 4p - 4a - 6b < 2p$ . Then  $(x_6, y_6) \in R_1$ . So we want  $x_6 < y_6$ , i.e.,  $a + 4b < p$  (a condition of  $R_3$ ).

We can repeat the above construction and gradually shrink  $\mathbf{B}$  inside  $X$  by removing regions  $R_1, R_2, R_3$ , and so on. By simple geometry we find that the total area left is equal to  $\frac{673}{2520}p^2$  which is about 6% larger than  $5/24p^2$ .

*Remark 5.4.* In weight three, we can show that for  $N = 1, 2, 3, 4$  there is a  $\mathbb{Z}$ -basis for  $\mathcal{MPV}(3, N)$  but we don't have any general results yet.

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