Motivic Complexes of Weight Three and Pairs of Simplices in Projective 3-Space

Jianqiang Zhao

Department of Mathematics, University of Pennsylvania, Philadelphia, Pennsylvania 19104
E-mail: jqz.math.upenn.edu

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0. INTRODUCTION

In the early 1990s Goncharov [14] proved Zagier’s conjecture about zeta functions at $s = 3$. His method is geometric in nature but also incorporates lots of information from algebra and analysis. In particular, for an arbitrary field $F$ he constructs the so-called weight $n$ ($n \geq 2$) polylogarithmic complex $(F(n); n, \delta)$,

\[
\begin{align*}
\mathcal{B}_n(F) & \xrightarrow{\delta} \mathcal{B}_{n-1}(F) \otimes F^\times \xrightarrow{\delta} \mathcal{B}_{n-2}(F) \otimes F^\times \xrightarrow{\delta} \cdots \\
n-2 & \delta \quad \mathcal{B}_2(F) \otimes \bigwedge^{n-2} F^\times \xrightarrow{\delta} \bigwedge^n F^\times
\end{align*}
\]

which is conjectured [14, pp. 54–55] to be quasi-isomorphic to the motivic complexes conjectured by Beilinson and Lichtenbaum. The group $\mathcal{B}_n(F)$ is
the quotient of $\mathbb{Z}[\mathcal{P}]$ by the subgroup $\mathcal{A}_j(F)$. This subgroup reflects (conjecturally all of) the functional equations of the single-valued polylogarithm $\mathcal{L}_n(z)$ discovered by Zagier [20] when $F = \mathbb{C}$ (see Eq. (1)). In the weight three case Goncharov gives some other versions of $\mathcal{A}_j(F)$ by explicitly defining the relation groups geometrically. All of these groups are conjectured to be isomorphic, at least modulo torsions. In this paper, we will prove this conjecture for the geometrically defined groups (Theorem 2.13).

The function $\mathcal{L}_n(z)$ is constructed from the classical polylogarithms which are defined as multi-valued functions $\text{Li}_n(z)$ on $\mathbb{C}\setminus [0, 1]$ by the iterated integral $\int_0^z \frac{dt}{t^2} \cdot \frac{dt}{t^2} \cdots \frac{dt}{t^2}$ in the sense of Chen [7]. When $z \in \mathbb{R}$ we can rewrite it as an $n$-dimensional integral

$$\text{Li}_n(z) = \int \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}.$$  

Aomoto [1] considered more general integrals where the integral $(dt_1/t_1) (dt_2/t_2) \cdots (dt_n/t_n)$ is over a real simplex in $\mathbb{C}P^n$. Let us recall the construction in some detail.

A simplex in the projective spaces $\mathbb{P}^n_F$ is an ordered set of hyperplanes $L = (L_0, \ldots, L_n)$. It is nondegenerate if the intersection of all the hyperplanes $L_i$ is empty. A face of $L$ is any nonempty intersection of the hyperplanes. A pair of simplices is admissible if they do not have common faces of the same dimension. A generic pair if all the faces of the two simplices are in general position. Given a nondegenerate simplex $L$ we may choose the coordinate system $[t_0, \ldots, t_n]$ in $\mathbb{P}^n_F$ such that $L_i = \{ t_i = 0 \}$ for $0 \leq i \leq n$. If $F = \mathbb{C}$ then we define the canonical differential form associated to $L$ as

$$\omega_L = d\log(t_1/t_0) \wedge \cdots \wedge d\log(t_n/t_0)$$

The Aomoto $n$-logarithm is a multi-valued function on configurations of nondegenerate admissible pairs of simplices $(L; M)$ in $\mathbb{C}P^n$ defined as

$$A_n(L; M) = \int_{\alpha_M} \omega_L,$$

where $\alpha_M$ is the $n$-cycle representing a generator of $H_n(\mathbb{C}P^n, M; \mathbb{Z})$.

Besides classical polylogarithms, there are other specializations of Aomoto polylogarithms. For example, Goncharov [16] defines multiple polylogarithms as

$$\text{Li}_{m_1, \ldots, m_k}(x_1, \ldots, x_l) = \sum_{0 \leq k_1 < k_2 \cdots < k_l} \frac{x_1^{k_1} \cdots x_l^{k_l}}{k_1^{m_1} k_2^{m_2} \cdots k_l^{m_l}}, \quad |x_i| < 1.$$
One can then use analytic continuation to extend the above to \( C \) as a multi-valued meromorphic function. We call \( l \) the \textit{length} and \( n_1 + \cdots + n_t \) the \textit{weight}. When the length \( l = 1 \) they are nothing but the classical polylogarithms.

In [12] Goncharov explicitly expresses the Aomoto trilogarithm by classical trilogarithms and products of polylogarithms of lower orders (dilogarithm and logarithm) and investigates the algebro-geometric structure lying behind it: different realizations of the weight three motivic complexes. We are able to complete this investigation in this paper by fully extending his ideas to study the double scissors congruence groups in greater detail.

The double scissors congruence groups are first introduced in [4] and then studied by Beilinson et al. in two papers [2, 3] from where the whole story begins. Briefly speaking the double scissors congruence group \( A_n \) is generated by admissible pairs of simplices, subject to a set of relations in \( P_\mathbb{F} \). (See Definition 1.1 for the detail.) The defining (double scissors) relations reflect (conjecturally all of) the functional equations of Aomoto polylogarithms. Furthermore, the graded object \( A_* \) actually forms a Hopf algebra with well-defined product \( \mu \) and coproduct \( \gamma \). They provide a bridge to the study of motivic polylogarithmic complexes.

According to Tannakian formalism the category \( \text{MTM}(F) \) of mixed Tate motives over a field \( F \) is supposed to be equivalent to the category of graded modules over a certain graded commutative Hopf algebra \( \mathcal{A} \) (see [4, 13, Chap. 3]). Therefore the Ext groups in the category \( \text{MTM}(F) \) of mixed Tate motives over \( \text{Spec}(F) \) are isomorphic to the cohomology of the Hopf algebra \( \mathcal{A} \). Beilinson et al. conjecture that \( A_* \) is isomorphic to \( \mathcal{A} \) and therefore the groups \( A_n \otimes \mathbb{Q} \) should have a Hopf algebra structure over \( \mathbb{Q} \). This is the primary motivation to study the groups \( A_n \).

On the other hand, by Beilinson’s conjecture, there is a negatively graded Lie algebra \( L_* (F) \) over \( \mathbb{Q} \) such that

\[
\text{Ext}^i_{\text{MTM}(F)} (\mathbb{Q}(0), \mathbb{Q}(n)) \cong H^i_{\text{mot}} (L_* (F)) \cong \text{gr}^*_{\gamma} \mathcal{K}_{2n-i}(F) \otimes \mathbb{Q}.
\]

A lot of evidence [15] shows that \( B_n (F) \otimes \mathbb{Q} \) (for \( n = 1, 2, 3 \)) is dual to the motivic Lie algebra \( L_{-n}(F) \) (the dual is between the ind and pro \( \mathbb{Q} \)-vector spaces). It follows from this line of thought [12] that the following conjecture should be true.

**Conjecture 0.1.** Let \( \Pi_n = \bigoplus_{j=1}^{n-1} \mu(A_j \otimes A_{n-j}) \) be the subgroup of prisms of \( A_n \). Then for \( n = 1, 2, 3 \)

\[
(A_n/\Pi_n) \otimes \mathbb{Q} \cong L_{-n}(F)^* \cong \mathcal{B}_n(F) \otimes \mathbb{Q}.
\]
This means that the dual to the Hopf algebra $A_v \otimes \mathbb{Q}$ is isomorphic to the universal enveloping algebra of the Lie algebra $L_v(F)$. This conjecture is trivial when $n=1$ (all are isomorphic to $F^* \otimes \mathbb{Q}$). For $n=2$ the first isomorphism is proved in [2] by taking $L_{-2}(F)$ to be the Bloch group $B_3(F)$ which is the quotient group of $\mathbb{Z}[\mathbb{P}^1_F]$ modulo the five-term relations.\footnote{In the literature the Bloch group sometimes refers to the subgroup of $B_3(F)$ which is isomorphic to the indecomposable part of $K_3(F)$ modulo torsions.} It is known that for number fields $B_3(F) \cong B_3(F)$.

Similar to $B_3(F)$ the group $B_4(F)$ is defined by Goncharov [12] as the quotient of $\mathbb{Z}[\mathbb{P}^1_F]$ by the subgroup $R_3(F)$ where $R_3(F)$ is generated by the generic seven-term relations of the generalized cross ratio $r_3$ and the Kummer–Spence relations (see Subsection 2.4). We call $B_3(F)$ the Bloch group of weight three.

The main purpose of this paper is to prove

**Main Theorem 0.2 (Main Theorem 3.1).** Let $F$ be a field. Modulo torsions one has

$$A_3(F) / \Pi_3(F) \cong B_3(F).$$

This result strongly supports the Conjecture 0.1. Taking $F = \mathbb{C}$ we can recover the result about the relation between Aomoto trilogarithm and the classical trilogarithm discovered in [12]:

**Corollary 0.3 (Corollary 3.29).** The Aomoto trilogarithms may be represented by classical trilogarithms, products of classical dilogarithm with logarithm, and products of logarithms.

The proof of the theorem further implies that the functional equations of classical trilogarithm can be obtained from the double scissors relations for the Aomoto trilogarithm.

The Aomoto trilogarithm is defined on a pair of quadruple hyperplanes in $\mathbb{C}P^3$ while the classical trilogarithm is defined on $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$. It is rather nontrivial problem to connect these two domains. In [12] Goncharov first discovered the map $a_3$ from the generic part $A^g_3$ of $A_3$ (the group generated by generic pairs of simplices modulo some relations similar to those of $A_3$) to $B_3$. It was in the process to generalize $a_3$ to $A_3$ that the author found a proof of the Main Theorem.

We now give an outline of this paper. In Sections 1 and 2 we provide a brief review of the double scissors congruence groups and four candidates of the groups $L_{-3}(F)^\vee$ among which three are defined geometrically and one universally. The only new thing here is the proof (Theorem 2.13) that
the three geometrically defined candidates for $L_{-3}(F)^c$ are isomorphic to each other modulo torsions.

All the key results of this paper are presented in Section 3. We treat both weight two and weight three cases even though weight two case has been dealt with in [2, 3]. We hope to give concise proofs of some of the results in weight two case and then use them as suggestions to us on how to prove things in weight three. It is in this section that we show our Main Theorem 0.2. Here we list some of the highlights:

1. After recalling the definitions of the maps $a_n: A_n \to B_n$ for $n = 2, 3$ in the first subsection we show that they are actually defined on $A_n/\Pi_n$.

2. We then define the map $\lambda_n: \mathbb{Z}[\mathbb{P}_1^1] \to A_n(F)/\Pi_n$ such that $(-1)^n \lambda_n(x) = A_n(x)$ is the $n$-logarithmic pair of simplices which corresponds to the classical $n$-logarithm if $F = \mathbb{C}$. In order to see that $\lambda_3$ factors through $B_3(F) = \mathbb{Z}[\mathbb{P}_1^1]/R_3(F)$ we need some involved geometric computations (Subsection 3.4) which relate all the multiple polylogarithmic pairs of weight three to the trilogarithmic pairs.

3. We finally show that the induced map $l_n: B_n(F) \to A_n(F)/\Pi_n$ from $\lambda_n$ is both injective (Subsection 3.6) and surjective (Subsection 3.7). The surjectivity of $l_3$ means that every admissible pair in $A_3$ has a decomposition into trilogarithmic pairs modulo prisms.

All the above steps in Section 3 can be carried out for the universally defined group $\mathfrak{R}_3(F)$ except that we don’t know whether the map $\mathfrak{R}_3(F) \to A_3/\Pi_3$ is well defined. We believe it is and state it as

**Conjecture 0.4.** The map $\lambda_3: \mathbb{Z}[\mathbb{P}_1^1] \to A_3/\Pi_3$, $\lambda_3(0) = \lambda_3(\infty) = 0$, and $\lambda_3(x) = -A_3(x)$ mod $\Pi_3$, sends $\mathfrak{R}_3(F)$ to zero.

We show that $\lambda_3(x) + \lambda_3(1-x) + \lambda_3(1-x^{-1}) = \lambda_3(1)$ and $\lambda_3$ sends Goncharov's 22-term relations $R_3(a, b, c)$ for the single-valued trilogarithm $\mathcal{L}_3(z)$ to zero (Proposition 3.15) where

$$R_3(a, b, c) = \{-abc\} + \bigoplus_{\text{cycle}} \left\{ \{ca-a+1\} + \{ ca-a+1 \over ca \} + \{c\} - \{1\} \right\}$$

$$+ \left\{ \{a(bc-c+1)\} + \{bc-c+1\} \over ca-a+1 \right\} + \left\{ \{ ca-a-1 \} - \{ca-a-1\} \over c \right\}$$

$$- \left\{ \{bc-c+1\} \over bc(ca-a+1) \right\}.$$ 

Therefore Conjecture 0.4 follows from the following

**Conjecture 0.5.** Modulo torsions the group $\mathfrak{R}_3(F)$ can be generated by $R_3(a, b, c)$ and $\{a\} + \{1-a\} + \{1-a^{-1}\} - \{1\}$ for $a, b, c \in F$. 


In fact we expect that \(\{a\} + \{1 - a\} + \{1 - a^{-1}\} - \{1\}\) and the 22-term relations generate all the functional equations of the trilogarithm \(\mathcal{L}(z)\).

As a byproduct, in the last section we are able to prove the following theorem which generalizes a result of Goncharov in [12] where he replaced \(A_3\) by the abelian group \(\mathbb{A}_3^0\) generated by generic pairs in \(\mathbb{P}_F^3\).

**Theorem 0.6 (Theorem 4.1).** Let \(F\) be a field, \(A_n = A_n(F)\) and \(B_n = B_n(F)\). Then the following diagram is commutative,

\[
\begin{array}{c}
A_3 \xrightarrow{\nu_2 \otimes \nu_1} (A_2 \otimes A_1) \oplus (A_1 \otimes A_2) \xrightarrow{\nu_1 \otimes \text{id} - \text{id} \otimes \nu_1} A_1 \otimes A_1 \otimes A_1 \\
B_3 \xrightarrow{\delta_3} B_2 \otimes F^\times \xrightarrow{\delta_2 \otimes \text{id}} \bigwedge^3 F^\times,
\end{array}
\]

where \(r\) is the cross ratio and \(\tau(x \otimes y) = y \otimes x\).

Little of the above work has been generalized to the weight four case. The difficulty comes from several aspects. First, the map \(a_4: A_4 \to \mathcal{A}_4\) is still an unlocked mystery. Second, even if \(a_4\) is defined the relation group \(\mathcal{R}_4\) is hard to play with. The existence of the analog of the Bloch groups \(B_4\) and \(B_3\) in the weight four case is unknown. One of the key reasons is that only some special functional equations for the tetralogarithms have been found (see [9, 10]). From the experience in lower weight cases the most general functional equation for \(\mathcal{L}(z)\) should have at least four variables such that all but a finitely many of the other functional equations are specializations of this one. The third problem in the weight four case grows out of geometric considerations. We know that there exist pairs of simplices in \(A_4\) in non-generic position whose inclusion relations cannot be found in any multiple polylogarithmic pairs of weight four or products of polylogarithmic pairs of lower weight (see [22, Chap. 6]). Therefore, the map \(\lambda_4: \mathbb{Z}[\mathbb{P}_F^4] \to A_4/\Pi_4, \lambda_4(x) = A_4(x)\), definitely is not surjective. However, it is believed [14, p. 244] that \(A_4/\Pi_4\) is isomorphic to \(\mathcal{A}_4 \otimes \bigwedge^3 \mathcal{B}_2\). As pointed out by Goncharov, the existence of the canonical embedding of \(\bigwedge^3 \mathcal{B}_2\) is a very intriguing problem.

Often in the paper, we disregard all the torsions and use the same notation to denote the corresponding objects tensored with \(\mathbb{Q}\).

## 1. DOUBLE SCISSORS CONGRUENCE GROUPS AND POLYLOGARITHMS

We shall begin with some notation. Let \(F\) be an arbitrary field. A *simplex* in the projective spaces \(\mathbb{P}_F^5\) is an ordered set of hyperplanes \(L = (L_0, ..., L_n)\).
It is nondegenerate if the intersection of all the hyperplanes $L_i$ is empty. A face of $L$ is any nonempty intersection of the hyperplanes. A pair of simplices is admissible if they do not have common faces of the same dimension. It is a generic pair if all the faces of the two simplices are in general position.

We now define the double scissors congruence groups $A_d(F)$ first introduced in [4] and modified in [3]. I add the trivial intersection axiom for the sake of completeness and simplicity.

**Definition 1.1.** Define $A_d(F) = \mathbb{Z}$. If $n > 0$ then $A_d(F)$ is the abelian group generated by admissible pairs of $n$-simplices $(L; M)$ subject to the following relations:

(R1) **Nondegeneracy.** $(L; M) = 0$ if and only if $L$ or $M$ is degenerate.

(R2) **Trivial intersection.** Suppose $L_0, ..., L_n$ and $M_0, ..., M_n$ are hyperplanes in $\mathbb{P}^{n+1}$. If $N = L_i$ or $N = M_i$ for some $i$ then

$$(N | L; M) := ((L_0 \cap N, ..., L_n \cap N); (M_0 \cap N, ..., M_n \cap N)) = 0.$$

(R3) **Skew symmetry.** For every permutation $\sigma$ of $\{0, ..., n\}$

$$\sigma L; M = (L; \sigma M) = \text{sgn}(\sigma)(L; M),$$

where $\sigma L = (L_{\sigma(0)}, ..., L_{\sigma(n)}).

(R4) **Additivity in $L$ and $M$.** For any $n + 2$ hyperplanes $L_0, ..., L_{n+1}$ and $n$-simplex $M$ in $\mathbb{P}^n$

$$\sum_{j=0}^{n+1} (-1)^j ((L_0, ..., \widehat{L_j}, ..., L_{n+1}); M) = 0$$

if every pair $((L_0, ..., \widehat{L_j}, ..., L_{n+1}); M)$ is admissible. A similar relation is satisfied by $M$.

(R5) **Projective invariance.** For every $g \in \text{PGL}_{n+1}(F)$

$$(gL; gM) = (L; M).$$

Denote by $[L; M]$ the class of $(L; M)$ in $A_d(F)$.

Recall the cross ratio on the projective line $\mathbb{P}^1_F$:

$$r(a, b, c, d) := \frac{(a - c)(b - d)}{(a - d)(b - c)}.$$
normalized by \( r(\infty, 0, 1, z) = z \). This provides us a convenient identification \( A_1 \cong F^p \).

The graded objects \( A_\ast \) should form a Hopf algebra with well-defined product \( \mu \) and coproduct \( \Delta \). We refer the interested reader to \([21, 23]\) for this and \([21, \text{Sect. 2}]\) for a detailed analysis of the above definition. There, we proved the following:

**Proposition 1.2 (Intersection Additivity).** For \( n \) hyperplanes \( M_1, \ldots, M_n \) and \( n+1 \) hyperplanes \( L_0, \ldots, L_n \) in \( \mathbb{P}^n_F \):

\[
\sum_{i=0}^n (-1)^i \left( \frac{L_i}{(L_0, \ldots, \hat{L}_i, \ldots, L_n)}; M \right) = 0
\]

if every pair \((L_i, (L_0, \ldots, \hat{L}_i, \ldots, L_n); M)\) is admissible on \( L_i \cong \mathbb{P}^{n-1}_F \). A similar relation holds for \( M \).

In this paper we will need the coproduct on \( A_3 \) whose detailed definition can also be found in \([22]\). For example, if an admissible pair \((L, M)\) in \( \mathbb{P}^n_F \) is in general position then we define

\[
\sigma_{n-k, k}([L; M]) = \sum_{I,J} \text{sgn}(I) \text{sgn}(J) [L_I | L_0, J; M_0, J] \otimes [M_J | L_0, I; M_0, J],
\]

where the index sets \( I, J \) run through \([1, \ldots, n] \) with \(|I| = k, |J| = n-k\), \( I = \{1, \ldots, n\} \setminus I \), \( \text{sgn}(I) = \text{sgn}(I, J) \) and similarly for \( J \). By convention, \( L_I \) appearing before the vertical bar means the intersection of all hyperplanes \( L_i \) with \( i \in I \) while those appearing after the vertical bar means simply the ordered set of hyperplanes.

Now we introduce an important function on hyperplanes in projective spaces. Let \( H_0, \ldots, H_n \) be \( n+1 \) hyperplanes in \( \mathbb{P}^n_F \) given by the equations

\[
H_i = \{ \sum_{j=0}^n a_{ij} t_j = 0 \}
\]

where \([t_0, \ldots, t_n] \) is the coordinate system in \( \mathbb{P}^n_F \). Then we set

\[
\Delta(H_0, \ldots, H_n) = \text{det}(a_{ij}).
\]

Although it is not well-defined because it depends on the choice of the equations of \( H_i \)'s we will see that this ambiguity does not matter every time this function appears in this paper. For example, for four points \( x_0, x_1, x_2, x_3 \in \mathbb{P}^1_F \), the cross ratio

\[
r(x_0, x_1, x_2, x_3) = \frac{\Delta(x_0, x_2) \Delta(x_1, x_3)}{\Delta(x_0, x_3) \Delta(x_1, x_2)}
\]

is independent of the choice of equations for \( x_i \)'s.
By the above definition, Proposition 2.3 of \cite{12} becomes

**Lemma 1.3 (Goncharov).** Under the identification $r: A_1 \simeq F^*$ by the cross ratio we have

\[

\nu_{n-1}(L; M) = - \sum_{i,j=0}^{n} (-1)^{i+j} A(M_j, L_i^-) \otimes [M_j \mid L_i^- \mid M_j^-];
\]

\[

\nu_{n-1,1}(L; M) = - \sum_{i,j=0}^{n} (-1)^{i+j} [L_i \mid L_i^- \mid M_j^-] \otimes A(L_i, M_j^-),
\]

where $L_i^-$ is the ordered set $(L_0, \ldots, L_i, \ldots, L_n)$ and similarly for $M_j^-$. By additivity and Proposition 1.2 the right hand side of the above formulas does not depend on the choice of equations of the faces of $L$ and $M$.

**Remark 1.4.** We only need this lemma when $n = 2$ and 3 which can be checked directly.

**Definition 1.5.** The $n$-simplex $L$ whose faces are $L_i = \{ t_i = 0 \}$ ($0 \leq i \leq n$) is called the standard (coordinate) simplex. For an arbitrary field $F$ we denote by $A_{n_1, \ldots, n_l}$ the subgroup of $A_n$ ($n = n_1 + \cdots + n_l$) generated by all multiple polylogarithmic pairs $A_{n_1, \ldots, n_l}(x_1, \ldots, x_l)$ corresponding to the multiple polylogarithms (all possible $x_1, \ldots, x_l$) and write $A(n)$ for the group generated by of $A_{n_1, \ldots, n_l}$ with $n_1 + \cdots + n_l = n$. Specifically, we can represent $A_{n_1, \ldots, n_l}(x_1, \ldots, x_l)$ by $(L; M)$ where $L$ is the standard simplex in $\mathbb{P}^n_F$ and $M$ is determined as follows: Take $a_i = 1/(x_i \cdots x_l)$ and the vertex facing $M_0$ to be

\[

m_0 = \begin{bmatrix} \bar{z}_0, \ldots, \bar{z}_n \end{bmatrix} = \begin{bmatrix} 1, -a_1, 0, \ldots, 0, -a_2, 0, \ldots, 0, \ldots, -a_l, 0, \ldots, 0 \end{bmatrix}_{n_1-1 \text{ times}}, \begin{bmatrix} 0, \ldots, 0 \end{bmatrix}_{n_2-1 \text{ times}}, \begin{bmatrix} 0, \ldots, 0 \end{bmatrix}_{n_l-1 \text{ times}}.
\]

The vertex $m_j$ ($1 \leq i \leq n$) facing $M_i$ is given by

\[

m_j = \begin{bmatrix} 1, \bar{z}_1, \ldots, \bar{z}_{n-1}, \bar{z}_{n-j+1} + 1, \ldots, \bar{z}_n + 1 \end{bmatrix}.
\]

**Remark 1.6.** If $F = \mathbb{C}$ and $M = A_{n_1, \ldots, n_l}(x_1, \ldots, x_l)$ a real simplex then the integral

\[

\int_{A_M^*} d\log(t_1/t_0) \wedge \cdots \wedge d\log(t_n/t_0)
\]
over the convex region bounded by the faces of $M$ corresponds to the multi-valued multiple polylogarithm

\[ (-1)^t \text{Li}_{n_1, \ldots, n_l}(x_1, \ldots, x_l). \]

This is the origin of the name *multiple polylogarithmic pair*.

For arbitrary fields, the polylogarithmic pair $A_n(t)$ will play a very important role in the rest of the paper. It has another realization (see [14, pp. 242–244]). We can define another simplex $M^{(n)}(x) = (M_0, \ldots, M_{n-1}, M_n(x))$ by

\[
M_0 : t_0 = t_1, \quad M_1 : t_0 - t_1 = t_2, \quad M_i : t_i = t_{i+1} \quad \text{for} \quad 2 \leq i \leq n-1, \quad M_n : t_n = xt_0.
\]

It is easy to verify that for the standard $n$-simplex $L$

\[ A_n(x) = (-1)^{(n+1)/2} [L; M^{(n)}(x)]. \]

In particular, $A_2(x) = -[L; M^{(2)}(x)]$ and $A_3(x) = [L; M^{(2)}(x)]$.

When $F = \mathbb{C}$, associated to the standard $n$-simplex $L$ is the canonical differential $n$-form $\omega_L = d \log(t_1/t_0) \wedge \cdots \wedge d \log(t_n/t_0)$. Choose a real simplex $M$ such that $(L; M)$ is an admissible pair and let $A_M$ be the $n$-cycle representing a generator of the relative cohomology group

\[ H_n(CP^n, \mathbb{C}; M; \mathbb{Z}) \]

with $\mathbb{Z}$-coefficients. Then $(L; M)$ gives rise to the Aomoto $n$-logarithm $A_n(t)$ via iterated integrals in the sense of K.-T. Chen [7]. D. Zagier further defines the single-valued version of $\text{Li}_n(t)$ on $\mathbb{C}$ by using Ramakrishnan’s result [18]:

\[
\text{Li}_n(z) = \begin{cases} 
\log |z| & \text{if } n = 1, \\
\Re(n \text{ : odd}) \left( \sum_{k=0}^{n-1} \frac{2k B_k}{k!} \log^k |z| \cdot \text{Li}_{n-k}(z) \right) & \text{if } n \geq 2,
\end{cases}
\]

(1)

where $B_k$ are Bernoulli numbers defined by $x/(e^x - 1) = \sum_{k \geq 0} B_k x^k/k!$.

2. DIFFERENT VERSIONS OF POLYLOGARITHMIC COMPLEXES

The goal in this section is twofold. First, we shall briefly review the definitions of different candidates for $L_{-\delta}(F)^+$ and then prove them to be isomorphic to each other except for the universally defined group $\delta(F)$. Second, we would like to extract some crucial results from quite a few
papers of Goncharov and investigate their relations. We have to mention the corresponding objects in weights one and two because weight three is built up from the lower weights.

2.1. The $\mathcal{B}$-Groups: Iterative Universal Definition

Let $\mathbb{Z}[P^1_F]$ be the free abelian group generated by $\{x\}$ where $x \in P^1_F$. According to [14, Sect. 4], we can define a family of quotient groups $\mathcal{B}_n(F)$ such that the relations in $\mathcal{B}_n(C)$ reflect functional equations of $L_n(z)$.

**Definition 2.1.** Set

$$\mathcal{B}_1(F) := \langle \{x\} + \{y\} - \{xy\} : x, y \in F^*; \{0\} ; \{\infty\} \rangle$$

and $\mathcal{B}_1(F) = \mathbb{Z}[P^1_F] / \mathcal{B}_1(F) \cong F^*$. For $n \geq 2$ we define the groups $\mathcal{B}_n(F)$ and $\mathcal{B}_{n-1}(F)$ inductively as follows. Assume now $\mathcal{B}_{n-1}(F)$ is already defined. Consider the homomorphisms

$$\delta_n(\mathcal{B}_F) : \mathbb{Z}[P^1_F] \to \begin{cases} F^* \land F^* & \text{if } n = 2 \\ \mathcal{B}_{n-1}(F) \otimes F^* & \text{if } n \geq 3 \end{cases}$$

$$\{x\} \mapsto \begin{cases} (1-x) \land x & \text{if } n = 2 \\ \{x\} \otimes x & \text{if } n \geq 3 \end{cases}$$

$$\{0\}, \{1\}, \{\infty\} \mapsto 0,$$

where $\{x\}_{n-1}$ is the image of $\{x\}$ in $\mathcal{B}_{n-1}(F) = \mathbb{Z}[P^1_F] / \mathcal{B}_{n-1}(F)$. Then we define the groups $\mathcal{B}_n(F)$ to be the subgroup of $\mathbb{Z}[P^1_F]$ generated by $\{0\}$, $\{\infty\}$ and

$$\{z(u) - z(u') : u, u' \in P^1_F, z = \sum_{\text{finite}} n_i \{f_i\} \in \ker \delta_n(F(t)), f_i \in F(t)\}$$

and set $\mathcal{B}_n(F) := \mathbb{Z}[P^1_F] / \mathcal{B}_n(F)$. We now can define the polylogarithm complex $\Gamma(F, n)$, of weight $n$ from $\mathcal{B}$-groups as

$$\mathcal{B}_n(F) \xrightarrow{\delta} \mathcal{B}_{n-1}(F) \otimes F^* \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{B}_2(F) \otimes \bigwedge^{n-2} F^* \xrightarrow{\delta} \bigwedge^n F^*,$$

where

$$\delta(\{x\}_k \otimes (g_1 \land \cdots \land g_{n-k}))$$

$$= \begin{cases} \{x\}_{k-1} \otimes (x \land g_1 \land \cdots \land g_{n-k}) & \text{if } k > 2, \\ (1-x) \land x \land g_1 \land \cdots \land g_{n-k} & \text{if } k = 2. \end{cases}$$
Remark 2.2. The above definition makes sense by [14, Lemma 1.16]. One can also define another family of groups $\mathcal{B}_d(F)$ by replacing $\mathbb{P}^1_F$ (resp. $F(t)$) in the above definition of $\mathcal{R}_d(F)$ by all smooth connected curves $X$ over $F$ (resp. $F(X)$). The following rigidity conjecture (see [14, Remark, p. 226; 15, p. 48]) is equivalent to the conjecture $\mathcal{B}_d(F) \cong \mathcal{B}_d(F)$. The above definition was given in [15] while $\mathcal{B}_d(F)$ were used in [14].

Rigidity Conjecture 2.3 (Beilinson). Let $F_0$ be the algebraic closure of the prime field of $F$. The canonical map $K^*_n(F_0) \to K^*_n(F)$ induces an isomorphism
\[
\text{gr}_n^* K_{2n-1}(F_0) \cong \text{gr}_n^* K_{2n-1}(F), \quad n \geq 2.
\]

2.2. The $\mathcal{B}$-Groups: Configurations of Points

Using geometry we can define another family of groups $\mathcal{B}_d(F)$. But only for $n \leq 3$ we know exactly the right definition of $\mathcal{B}_d(F)$ to make it a candidate for $L_n(F)$. For $n > 3$ we believe that our definition is too large and it should have a strict quotient group which is isomorphic to $L_n(F)$.

Definition 2.4. Let $\mathcal{C}_{n,m}(F)$ be the free abelian group generated by all possible configurations (meaning modulo action of $\text{PGL}_{n,m}(F)$) of $n$ points in $\mathbb{P}^m_F$. Set
\[
\mathcal{B}_n(F) := \mathbb{Z}[\mathcal{C}_{n,m}(n-1)]/\mathcal{R}_n(F),
\]
where the relations $\mathcal{R}_n(F)$ are as follows. For $n = 2$ and $3$ we define
\[
L_n : \mathbb{Z}[\mathbb{P}^1_F \setminus \{0, 1, \infty\}] \to \mathbb{Z}[\mathcal{C}_{n,m}(n)]
\]
\[
z \mapsto L_n(z)
\]
as follows. For any $(l_0, l_1, l_2, l_3) \in \mathcal{C}_4(1)$ such that $r(l_0, l_1, l_2, l_3) = z$, $L_2(z) = (l_0, l_1, l_2, l_3)$; for any $(x_1, x_2, x_3, y_1, y_2, y_3) \in \mathcal{C}_6(2)$, $y_i \in \mathbb{P}_{x_i} \mathbb{Z}$ (1 \leq i \leq 3, x_4 = x_1, see Fig. 1) such that $r(y_2 | x_1, x_2, y_1, y_3) = z$, we define $L_3(z) = (x_1, x_2, x_3, y_1, y_2, y_3)$. (The $L_n$'s are not well defined, but modulo the relation groups to be defined below, they are.) When the $y_i$'s are collinear in the above we denote the configuration by $\eta_3$. The relation group
\[
\mathcal{R}_3(F) := \left\{ \sum_{j=0}^{4} (-1)^j (l_0, ..., l_j, ..., l_4) : l_0, ..., l_4 \in \mathbb{P}_F^1 \right\}
\]
so-called five-term relations), and $\mathfrak{R}_d(F)$ is generated by

(i) \{$(l_0, \ldots, l_5)$: two of the $l_i$'s coincide or four collinear}\;

(ii) The seven-term relations:

$$\left\{ \sum_{i=0}^{6} (-1)^i (l_0, \ldots, \widehat{l}_i, \ldots, l_5) : \forall l_0, \ldots, l_5 \in \mathbb{P}_F^2 \right\};$$

(iii) (Setting $L_3'(x) = L_3(x) + 2L_3(1-x) - \eta_3$)

$$\left\{ \sum_{i=0}^{4} (-1)^i L_3'(r(l_i | l_0, \ldots, \widehat{l}_i, \ldots, l_4)) : l_0, \ldots, l_5 \in \mathbb{P}_F^2, l_2 = \overline{l_0 l_1} \cap \overline{l_3 l_4} \text{ and } l_5 \text{ is in general position (see Fig. 2)} \right\};$$

For $n > 3$ we can set $\mathfrak{R}_d(F)$ to be the group generated by:

(i) Degeneracy: $(l_0, \ldots, l_2n)$ for any $l_0, \ldots, l_2n \in \mathbb{P}_F^n$ such that there are $2k + 2$ points lie in some $k$-dimension plane;

(ii) $(2n+1)$-term relations: For any $l_0, \ldots, l_{2n} \in \mathbb{P}_F^n$

$$\sum_{j=0}^{2n} (-1)^j (l_0, \ldots, \widehat{l_j}, \ldots, l_{2n}).$$

Remark 2.5. The skewsymmetry can be derived from (i) with $k=0$ and (ii): For any permutation $\sigma$ of $\{1, \ldots, 2n\}$ and $2n$ points $l_1, \ldots, l_{2n} \in \mathbb{P}_F^n$

$$(l_1, \ldots, l_{2n}) = \text{sgn}(\sigma)(l_{\sigma(1)}, \ldots, l_{\sigma(2n)}).$$

Notice that the correct definition of $\mathfrak{B}_d(F)$ when $n > 3$ is unknown because of the mysterious special relation (iii) in $\mathfrak{R}_d(F)$ which is independent of (i) and (ii). (See [14, pp. 213–214]; note, however, there is a typo on line 6 on p. 214. It should read $\ell(m_0, \ldots, m_5) = \text{Alt}(m_0, m_1, m_3, m_4, m_5) \neq 0$.}

**FIG. 1.** $r(y_3 | x_1, x_2, y_1, y_2) = z.$
2.3. The $\tilde{B}$-Groups: Explicit Definition Using Cross Ratio

By using cross ratio one can define a third family of groups which are closely related to the above groups $\mathfrak{B}(F)$ and $\mathfrak{B}_q(F)$.

**Definition 2.6.** Let $\tilde{B}_n(F) = \mathbb{Z} [\mathbb{P}^1] / \tilde{R}_n(F)$ where the groups $\tilde{R}_n(F)$ are defined as

$$\tilde{R}_1(F) = \left\langle \{ \infty \}, \{ xy \} - \{ x \} - \{ y \} : x, y \in F \right\rangle$$

$$\tilde{R}_2(F) = \left\{ \{ 0 \}, \{ 1 \}, \{ \infty \}, \sum_{i=0}^{4} (-1)^i \{ r(x_i, ..., x_{i+1}) : x_0, ..., x_4 \in \mathbb{P}^1 \} \right\},$$

where we set the cross ratio $\{ r(a, b, c, d) \} = \{ 0 \}$ if any two of $a, b, c, d$ are the same, and $\tilde{R}_3(F)$ is generated by

(i) $\{ x \} + \{ 1 - x \} + \{ 1 - x^{-1} \} - \{ 1 \} : x, y \in F \setminus \{ 1 \}$;

(ii) $R_3(l_0, ..., l_5, z) = \left\{ l_0, ..., l_5, z \in \mathbb{P}^2 \text{ distinct}, \right.$

$$A(l_1, l_3) \neq 0, A(l_0, l_2, l_4) \neq 0, \left. l_{i+1} \in \overline{l_{2i}l_{2i+2}} (0 \leq i \leq 2, l_0 = l_6), \right.$$

$z$ in general position

where $R_3(l_0, ..., l_5, z)$ is defined as $[14, p. 281 (5.1)]$ using cross ratio (see Fig. 3.). Here $A(l_1, l_3) \neq 0$ means that $l_1$, $l_3$, and $l_5$ are not collinear. If

\[ l_3 \]

\[ l_5 \]

\[ l_5 \]

\[ l_0 \]

FIG. 3. $z$ in the general position.
We define $\mathcal{R}_n$ to be the subgroup of $\mathbb{Z}[\mathbb{C}^\times]$ generated by the functional equations of the $n$-logarithm $\mathcal{L}_n(z)$. By abuse of notation we let $\{x\}$ also denote the projection of $\{x\}$ to $\tilde{B}_n(F)$. Similar to the complex $\mathcal{I}(F;n)_*$ constructed from $\mathcal{B}_n(F)$ we can define the polylogarithmic complex of weight $n$ from $\tilde{B}_n(F)$ groups as

$$
\tilde{B}_n(F) \xrightarrow{\delta} \tilde{B}_{n-1}(F) \otimes F^* \xrightarrow{\delta} \ldots \xrightarrow{\delta} \tilde{B}_3(F) \otimes \bigwedge F^* \xrightarrow{\delta} \bigwedge F^*.
$$

It's well-defined for $n=2,3$ though we don't know this in general.

**Remark 2.7.** (1) Relation $\{x\} = \{x^{-1}\}$ can be obtained from (i) as follows. Let $I(x) = \{x\} + \{1-x\} + \{1-x^{-1}\} - \{1\}$. Then $\{x\} - \{x^{-1}\} = I(x) - I(x^{-1}) \in \tilde{R}_3(F)$.

(2) The last relation in the definition of the group $\tilde{B}_3(F)$ on [14, p. 293] is incorrect since $\{1\} \notin \tilde{R}_3(F)$ although $\{1\} \in \tilde{R}_3(F)$ (thus $\delta^3(\{1\}) = 0$). Note that our definition of $\tilde{R}_3(F)$ is defined modulo torsion (the prevailing assumption of this paper). Otherwise, we need to add $\{0\}$ and $\{\infty\}$ to $\tilde{R}_3(F)$ because we only know that $2\{0\} = I(1), 2\{\infty\} = I(\infty) \in \tilde{R}_3(F)$.

Historically, $\tilde{B}_2(F)$ is called the Bloch group which is introduced by Dupont and Sah [8] and Suslin. (However, there does not exist a unanimous agreement on the name. Sometimes in the literature the Bloch group refers to the subgroup of $\tilde{B}_2(F)$ which is isomorphic to the indecomposable part of $\mathcal{K}_3(F)$ modulo torsions.) Bloch shows that $\tilde{R}_2(\mathbb{C})$ is the group of functional equations of the dilogarithm $\mathcal{L}_2(z)$ (see [6, Theorem 7.4.5]). Through a geometric approach, Goncharov [14, Theorem 1.10, p. 215, proof in Sect. 9] proves that $\tilde{R}_3(\mathbb{C})$ is a subgroup of the functional equations for the trilogarithm $\mathcal{L}_3(z)$. He further conjectures [11, Conjecture 1, p. 158] that this actually is the full group. Therefore one hopes that $\tilde{R}_n(\mathbb{C})$ can be always defined as the functional equations of the $n$-logarithm $\mathcal{L}_n(z)$.

It is a very intriguing problem to determine explicitly the groups $\tilde{R}_n(F)$ for $n \geq 4$.

**Remark 2.8.** Because $\tilde{R}_n(F) \subset \mathcal{B}_n(F)$ for $n=2,3$ (see [14, p. 225, 3rd line]), we see that there are surjection maps $\beta: \tilde{B}_n(F) \to \mathcal{B}_n(F)$. This is why we say $\mathcal{B}_n(F)$ is universally defined because we only know it is the smallest candidate for $L_{n-1}(F)$ at present.

It is trivial to see that $\tilde{B}_2(F) \cong \mathcal{B}_2(F)$ by $L_2$ and the cross ratio map. Further, Goncharov establishes the following theorem...
Theorem 2.9 [14, Theorem A, p. 293; 11, Theorem 4, p. 160]. Modulo 6-torsions, the homomorphism $L_3$ induces an isomorphism

$$L_3 : \mathcal{B}_3(F) \cong \mathcal{B}_3(F).$$

Remark 2.10. The proof of the theorem is rather complicated and ingenious. Goncharov first proves that $L_3$ is well defined [14, Theorem 5.1, p. 281] and surjective [14, Theorem 4.14, p. 279]. Then he constructs a homomorphism $M_3 : \mathcal{B}_3(F) \to \mathcal{B}_3(F)$ such that $L_3 \cdot M_3 = \text{id}$ [14, p. 286] which proves the injectivity of $M_3$. By definition it is easy to see that $M_3$ is also surjective. Note, however, the definition $M_3$ is not skew-symmetric. One should use the generalized cross ratio $r_3$ to be defined in next section where we shall show that, up to a constant, it is skew-symmetrization of $M_3$.

2.4. Bloch Group $B_3(F)$ and Generalized Cross Ratio

We first recall the generalized cross ratio $r_3$ appearing first in [15]. For six hyperplanes in $\mathbb{P}^2_F$ we put

$$r_3(H_1, \ldots, H_6) := \frac{1}{15} \text{Alt}(1, \ldots, 6) \left[ \frac{\text{Alt}(H_1, H_2, H_4) \text{Alt}(H_2, H_3, H_5) \text{Alt}(H_3, H_5, H_6)}{\text{Alt}(H_1, H_2, H_3) \text{Alt}(H_2, H_5, H_6) \text{Alt}(H_2, H_3, H_4)} \right] \in \mathbb{Q}[\mathbb{P}^1_F].$$

Here, we set $\{a/b\} = \{0\}$ if $a = b = 0$. We can similarly define $r_3(l_1, \ldots, l_6)$ for $l_i \in \mathbb{P}^2_F$.

Remark 2.11. Clearly, $r_3$ is well-defined just like the cross ratio. Nevertheless, it seems that we do not know if images of $r_3$ actually lie in $\mathbb{Z}[\mathbb{P}^1_F]$ although evidence shows that they do. Whether true or false it is of no importance in this paper because we do not care about the torsions here.

In [12] Goncharov provides a new version of the group $\tilde{B}_3(F)$ which we denote by $B_3(F)$ which is the quotient group $\mathbb{Q}[\mathbb{P}^1_F]/R_3(F)$. Here the relation group $R_3(F)$ is generated by the seven-term relations

$$\sum_{i=0}^{7} (-1)^i r_3(l_0, \ldots, \hat{l}_i, \ldots, l_6), \quad l_i \in \mathbb{P}^2_F,$$

where the points $l_0, \ldots, l_6$ are in general position, and the KS (Kummer-Spence) relations reflecting functional equation for the trilogarithm.
\[
\text{KS}(x, y) = -\left\{ \frac{x(1-y)^2}{y(1-x)^2} \right\} - \left\{ xy \right\} - \left\{ \frac{x}{y} \right\} - 2\{1\} \\
+ 2 \left\{ \frac{x(1-y)}{x-1} \right\} + \left\{ \frac{y(1-x)}{y-1} \right\} + \left\{ \frac{y(1-x)}{y-1} \right\} \\
+ \left\{ \frac{1-x}{1-y} \right\} + \left\{ y \right\} + \left\{ x \right\}.
\]

(2)

**Definition 2.12** For a field \( F \) we call \( B_3(F) \) the *Bloch group of weight three* defined over \( F \).

By the remark after the definition of the relation group \( R_3(F) \) in [12] we see that it can also be generated by the seven-term relations for arbitrary configurations of seven point in \( \mathbb{P}^2_F \). One may find that the relation \( \{x\} - \{x^{-1}\} \) is missing in our definition of \( R_3 \) when comparing it with Goncharov’s original definition. But we have the KS relations

\[
\text{KS}(1, x) = \{x\} - \{x^{-1}\} \quad \text{and} \\
\text{KS}(1-x, 0) = 2\{1 - x^{-1}\} + \{1-x\} + \{x\} - \{1\}.
\]

This is why we throw away the relation \( \{x\} - \{x^{-1}\} \).

We now have constructed four different versions of the Bloch group \( B_3 \). Are they all the same? The next theorem confirms this for all the geometrically defined versions.

In the proof of this theorem we will need some results from the next section so the interested readers can jump over the rest of this section in the first reading and come back later. In the proof we will also make use of the maps \( L_3 \) and \( M_3 \) appearing in Remark 2.10 so we first correct some misprints in the references. The definition of \( R_3(x_i, y_i; z) \) on [14, p. 205] is not the same as on [14, p. 285]. If we use the one first occurred then the equation in Lemma 5.2 of [14] should read

\[
L_3 R_3(x_i, y_i; z) = L_3 R_3(l_0, ..., l_5; z) + 3\eta_3,
\]

which implies that \( L_3 R_3(x_i, y_i; z) = 0 \) by Relation (ii) of Definition 2.6 since \( \eta_3 = L_3(\{1\}) \) by [14, Lemma 4.9].

**Theorem 2.13.** Let \( F \) be a field. Then modulo torsions

\[ B_3(F) \cong \mathfrak{B}_3(F) \cong \mathfrak{B}_3(F). \]

**Proof.** We may prove the theorem using the group \( \mathfrak{B}_3(F) \) as the go-between. Because \( M_3 : \mathfrak{B}_3(F) \cong \mathfrak{B}_3(F) \) (see Remark 2.10) it suffices to
prove that there is an isomorphism \( \tilde{r}_3 : \mathcal{B}_3(F) \approx B_3(F) \) induced by the generalized cross ratio.

First we notice that Relation (ii) of Definition 2.6 can be expressed by Goncharov's 22-term relation \( R_3(a, b, c) \) (see [14, p. 208]) which turns out to be 1/6 of the 7-term relation for the 7 points \((l_0, ..., l_5, z)\) (Lemma 3.13) we see that the group \( \tilde{R}_3(F) \) in Definition 2.6 is apparently smaller. Therefore we have

Fact (A). The group \( \tilde{R}_3(F) \) includes KS relations and

Fact (B). There is a surjective map \( s : \tilde{B}_3(F) \rightarrow B_3(F) \).

We first show that \( r_3 \) induces a map \( \tilde{r}_3 : \mathcal{B}_3(F) \rightarrow B_3(F) \) which coincides with \( s \) on \( M_3 \). Indeed, we only need to prove this on the generators of \( \mathcal{B}_3(F) \) (see [14, p. 286]). The computations below use the KS relations which are included in both \( R_3 \) and \( \tilde{R}_3 \). One should check this by looking at the proof of [12, Lemma 3.7] and direct calculation to see that no seven term relation is needed here.

Let \( \tilde{M}_3 \) be the map to \( \mathbb{Z}[[F^1]] \) corresponding to \( M_3 \).

(a) Without loss of generality, we may take

\[
\eta_3 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.
\]

Then from (b) below we see that \( r_3(\eta_3) = -6 \{1\} = -6 \tilde{M}_3(\eta_3) \) modulo KS relations.

(b) For a configuration as [14, Fig. 5.3a], we may assume

\[
[l_0, ..., l_5] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -x \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.
\]

By Corollary 3.7 we see that modulo KS relations

\[
r_3(l_0, ..., l_5) = -6 \{x\} = -6 \{ r(l_5 \mid l_0, l_1, l_2, l_3) \} = -6 \tilde{M}_3(l_0, l_1, l_2, l_3, l_4, l_5).
\]

(c) For a configuration as [14, Fig. 5.3b], we may assume

\[
(l_0, ..., l_5) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & a & 1 & 1 \\ 0 & 0 & 1 & 0 & b & 1 \end{bmatrix}.
\]
By definition (note the sign error in case (c) in [14, p. 286])

\[ 3M_3(l_0, \ldots, l_5) = \sum_{i=0}^{4} (-1)^i (1 + 2x) \cdot \{ r(l_5 | l_0, \ldots, \hat{l}_i, \ldots, l_5) \} + \{ 1 \}, \]

where \( \chi \{ a, b, c, d \} = \{ a, c, b, d \} \). Using relations \( \{ x \} - \{ x^{-1} \} \) and \( \{ 1 - x \} + \{ 1 - x^{-1} \} - \{ 1 \} \) a few times we see that modulo KS relations

\[
-3\vec{M}_3(l_0, \ldots, l_5) = \{ a \} - \left\{ \frac{a-1}{a} \right\} - \left\{ \frac{b-1}{ab} \right\} + \left\{ \frac{ab-b+1}{ab} \right\} - \{ 1-b \} \\
+ \{ b \} - \left\{ \frac{ab-b+1}{a} \right\} + \left\{ \frac{(1-a)(1-b)}{-a} \right\} \\
- \{ b-ab \} + \{ ab-b+1 \}. \tag{4}
\]

By direct computation using key Lemma 3.2 and Lemma 3.5 we have

modulo KS relations

\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & a & 1 & 1 \\
0 & 0 & 1 & 0 & b & 1
\end{bmatrix}
= \vec{r}_3 \begin{bmatrix}
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & a & 1 & 0 & 1 \\
0 & 0 & 0 & b & 1 & 1
\end{bmatrix}
= 2\lambda_3 \begin{bmatrix}
1 & 0 & 1 & 1 & b & 0 \\
0 & 1 & a & 1 & b-1 & 1
\end{bmatrix}
= 2\lambda_3 \begin{bmatrix}
1 & 0 & 1 & 1 & b-1 & 1 \\
0 & 1 & 1 & 0 & ab & a
\end{bmatrix}
= -6\vec{M}_3(l_0, \ldots, l_5). \tag{5}
\]

For all other configurations, we can use the seven-term relations which are satisfied by both \( M_3 \) and \( \vec{r}_3 \).

Now that \( \vec{r}_3 = s \cdot (-6M_3) \) is surjective we turn to show it is injective. We only need to construct an inverse map \( s_3 \): \( B_3(F) \to B_3(F) \). Clearly all KS relations are sent to zero by \( \lambda_3 \) [14, Theorem 5.1] from Fact (A). From the above we see that \( r_3 + 6\vec{M}_3 \) is zero modulo KS relations so \( (-L_3/6) \cdot r_3 = L_3 \cdot \vec{M}_3 = \text{id} \) modulo \( \mathcal{R}_3(F) \) from Remark 2.10. Thus any seven-term relation in \( \mathbb{Z}[[ F ]^+] \)

\[
\sum_{i=0}^{6} r_3(l_0, \ldots, \hat{l}_i, \ldots, l_5)
\]
is mapped to

\[ \sum_{i=0}^{6} (l_0, \ldots, \widehat{l_i}, \ldots, l_6) = 0 \pmod{\mathcal{R}_3(F)} \]

by \(-L_3/6\). This means that the map \((-L_3/6)\) induces a map \(\mathcal{R}_3(F) \rightarrow \mathcal{R}_3(F)\) (since \(\mathcal{R}_3(F)\) can be generated by all the seven-term relations) which is inverse to \(\mathcal{R}_3\).

This completes the proof of the theorem. \(\Box\)

To simplify notation, we will use \(B_2(F)\) to stand for \(\mathcal{B}_2(F)\) in the rest of the paper.

3. ISOMORPHISMS BETWEEN \(A_n/I_n\) AND GEOMETRIC CANDIDATES OF \(L_{-\omega}(F)^r\)

The goal of this section is to prove (always modulo torsions)

**Main Theorem 3.1.** Let \(F\) be a field. For \(n = 2, 3\) one has

\[ A_n(F)/I_n(F) \cong B_n(F). \]

Therefore

\[ A_n(F)/I_n(F) \cong B_n(F) \cong \mathcal{R}_n(F) \cong \mathcal{B}_n(F). \]

The proof is carried out in the next seven sections. Here we give the outline. First, we show that there are well-defined maps \(a_n: A_n/I_n \rightarrow B_n\) for \(n = 2, 3\) (Proposition 3.8). After that we construct well-defined maps \(l_n: B_n \rightarrow A_n/I_n\) (Theorem 3.9) such that \(a_n \circ l_n = \text{id}\) (Proposition 3.27) which imply the injectivity. The surjectivity of \(l_n\) is proved in Theorem 3.28. These establish the isomorphisms in the Main Theorem 3.1. The difficult part is to prove that all the maps above are well-defined.

3.1. The Maps \(a_n: A_n/I_n \rightarrow B_n, n = 2, 3\)

We first recall the definition of the maps \(a_n: A_n \rightarrow B_n\) for \(n = 2, 3\) given in [12] and then prove that \(a_n(I_n) = 0\). Originally \(a_2\) was first given in [2] and then clarified in [12].

Using the cross ratio we put

\[ a_2: A_2 \rightarrow B_2, \]

\[ [L; M] \mapsto \sum_{i,j=0}^{2} (-1)^{i+j} \{ r(L_0, \ldots, \widehat{L_i}, \ldots, L_2; M_0, \ldots, \widehat{M_j}, \ldots, M_2) \}. \]
To define \( a_3 \) we need two additional maps \( a'_3 \) and \( a''_3 \):

\[
a'_3([L; M]) = \sum_{i,j=0}^{3} (-1)^{i+j} r_3(L_i|L_0, ..., \hat{L}_i, ..., L_3; M_0, ..., \hat{M}_j, ..., M_3),
\]

where we use \( r_3 \) to denote its image in \( B_3 \) and

\[
a''_3([L; M]) = \sum_{i,j=0}^{3} (-1)^{i+j} \times \mu_3(L_i \cap M_j|L_0, ..., \hat{L}_i, ..., L_3; M_0, ..., \hat{M}_j, ..., M_3)
\]

where for six points \( x_1, x_2, x_3; y_1, y_2, y_3 \) on a line

\[
\mu_3(x_1, x_2, x_3; y_1, y_2, y_3) = \frac{1}{2} \text{Alt}_{[x_1, x_2, x_3], [y_1, y_2, y_3]} \{ r(x_1, y_2, x_2, y_1) \}.
\]

Finally we set

\[
a_3 = \frac{1}{6} a'_3 - \frac{1}{6} a''_3 : A_3 \to B_3.
\]

As pointed out in [12], neither \( a'_3 \) nor \( a''_3 \) is good enough for our purposes. Further, the key Lemma 3.7 in [12] provides us an extremely useful relation between \( r_3 \) and \( \mu_3 \) when we have a special degenerate configuration. However, we need to correct the sign in that lemma.

**Lemma 3.2.** Let \( (x_1, x_2, x_3; y_1, y_2, y_3) \) be six points in \( \mathbb{P}^2 \) where the first three points lie on a line \( X \) while at most one \( y_i \) is on \( X \) (see Fig. 4). Let \( n_i \) be the intersection of the line \( y_j y_k \) with \( X \) (1 \( \leq i, j, k \leq 3 \) are different). Then in \( B_3(F) \)

\[
r_3(x_1, x_2, x_3; y_1, y_2, y_3) = 2 \mu_3(x_1, x_2, x_3; n_1, n_2, n_3).
\]

**Remark 3.3.** The sign of \( \mu_3(\mathbb{C}(b, c)) \) in the proof of Lemma 3.7 in [12] is incorrect, so is the proof of Proposition 3.6. Actually, without using seven-term relations this lemma directly implies that if \( M \) is degenerate
then $a_3([L; M]) = 0$. All the other formulas in [12] are now correct without any further modifications.

Remark 3.4. In [12], $a_3$ was only defined for pairs of simplices $(L; M)$ in general position because the coproduct on $A_3$ was not defined for pairs in non-general position. But it is not too difficult to extend $a_3$ to $A_3$ once the coproduct on $A_3$ is defined.

The following lemmas will be used a few times later so we list them here for future reference.

**Lemma 3.5.** Let $D = ad - bc$ and $[t]_3 = \{t\}_3 - \{1 - t\}_3$. Then

$$
\mu_3 \left[ \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right] = \left[ \begin{array}{c} b \\ d \end{array} \right]_3 - \left[ \begin{array}{c} a \\ c \end{array} \right]_3 - \left[ \begin{array}{c} ad \\ D \end{array} \right]_3
$$

$$
- \left[ \begin{array}{c} b(a - c) \\ D \end{array} \right]_3 - \left[ \begin{array}{c} c(d - b) \\ D \end{array} \right]_3.
$$

**Lemma 3.6.** Let $I$ be the $3 \times 3$ identity matrix. For any $a, b \in F^*$ and $c, e, f \in F$ we have

$$
\mu_3 \left[ \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & a \end{array} \right] = -\mu_3 \left[ \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & b \end{array} \right] = \mu_3 \left[ \begin{array}{ccc} 0 & 1 & a \\ 0 & 1 & b \end{array} \right] = \tau \{1\}_3.
$$

**Corollary 3.7.** One has

$$
r_3 \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] = -6\{x\}_3.
$$

**Proof.** Easy calculation shows that

$$
r_3 \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] = -2(\{x\}_3 + \{1\}_3 - \{1 - x^{-1}\}_3) - \{1 - x\}_3 = -6\{x\}_3.$$

Proposition 3.8. For \( n = 2, 3 \), \( a_n(\Pi_n) = 0 \).

Proof. First we show that \( a_2(\Pi_2) = 0 \). Let \( L \) be the standard simplex: \( L_i = \{ t_i = 0 \} \), \( i = 0, 1, 2 \). Let \( M \) be the rectangle bounded by \( t_1 = t_0 \), \( t_2 = at_0 \) and \( t_2 = bt_0 \). We can cut \( M \) into two triangles \( N \) and \( Q \) along the diagonal \((a - b) t_0 + (a - 1) t_1 - (b - 1) t_2 = 0\) as shown in Fig. 5.

The idea to cut the rectangle is to make sure that the common edge (i.e., \( Q_0 = N_1 \)) should occur with different sign in \( N \) and \( Q \) if we assign \((-1)^i\) to the \( i \)th face of a triangle. Then we see that

\[
\begin{align*}
  a_2([L; N]) + a_2([L; Q]) &= [L_2 | L_{0, 2}; N_{0, 1}] - [L_1 | L_{0, 2}; N_{1, 2}] \\
  &= [L_1 | L_{0, 2}; Q_{0, 1}] - [L_2 | L_{0, 1}; Q_{0, 2}] \\
  &= \left\{ \begin{array}{l}
      a(b - 1) \\
      b - a
    \end{array} \right\}_2 - \left\{ \begin{array}{l}
      a - 1 \\
      a - b
    \end{array} \right\}_2 \\
  &= \left\{ \begin{array}{l}
      b(a - 1) \\
      a - b
    \end{array} \right\}_2 - \left\{ \begin{array}{l}
      b - 1 \\
      b - a
    \end{array} \right\}_2 = 0
\end{align*}
\]

because \( \{s\}_2 + \{1 - s\}_2 = 0 \) for any \( s \in \mathbb{R}(F) \) by taking the five-term relation on \((\infty, 0, 1, 0, s)\).

The proof that \( a_3(\Pi_3) = 0 \) is lengthy and computational in nature which will be given in the next section.

The above proposition implies that maps \( a_n \) can be defined on \( A_n/\Pi_n \) for \( n = 2, 3 \).

3.2. Proof of \( a_3(\Pi_3) = 0 \)

This section involves a lot of computation from which it is possible to trace back to a bunch of seven-term relations such that one can formulate a mostly geometric proof. However, this process is rather cumbersome and, in fact, unnecessary.
Throughout this section $L$ always denotes the standard coordinate simplex. Notice that $\Pi_3$ is generated by two kinds of geometric objects:

1. cubes $C(x, y, z) = \mu(x \otimes y \otimes z)$ for $x, y, z \neq 0, 1$,

2. prisms (in the literal sense) $P(x, y) = \mu(x \otimes A_2(y))$ for $x, y \neq 0$ and $x \neq 1$.

This is because $A_2$ is generated by rectangles and $A_2(y)$ for $y \neq 0$ (see the proof of Theorem 3.28). Here we have identified $A_1(F)$ with $F^*$ by the cross ratio.

As an easy start we want to deal with the cubes first (see Fig. 6). The vertices of the cube $C(x, y, z)$ in $\mathbb{P}_3$ are $A = [1, x, 1, 1], B = [1, x, y, 1], C = [1, 1, y, 1], D = [1, 1, 1, 1], E = [1, x, 1, z], F = [1, x, y, z], G = [1, 1, y, z]$ and $H = [1, 1, 1, z]$. We cut the cube into six simplices $M(i)$ ($1 \leq i \leq 6$) whose orientations are

$M^{(1)} = (ABD, ABE, ADF, BDF), \quad M^{(2)} = (AEF, ADF, BEF, ADE),$

$M^{(3)} = (DEH, DEF, EFH, DFH), \quad M^{(4)} = (BDF, BCD, BCF, CDF),$

$M^{(5)} = (DFG, CDG, CDF, CFG), \quad M^{(6)} = (DFH, FGH, DGF, DFG),$

where the equations of the faces $CDG, DEH, ABD, ABE, BCF, EFH, ADF, CDF,$ and $BDF$ are given by the columns of

$$
\begin{bmatrix}
1 & 1 & 1 & x & y & z & y-z & z-x & x-y \\
-1 & 0 & 0 & -1 & 0 & 0 & 1-z & y-1 & 0 \\
0 & -1 & 0 & 0 & -1 & -1 & z-1 & 0 & 1-x \\
0 & 0 & -1 & 0 & 0 & 1-y & x-1 & 0 & 0
\end{bmatrix} = 0,
$$

respectively. Notice that the orientations of $M^{(i)}$s are chosen in such a way that if we assign $(-1)^j$ to the $j$th face of a tetrahedron then the algebraic

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig6.png}
\caption{A cube $C(x, y, z)$, product of $x, y$ and $z$ in $A_1$.}
\end{figure}
sum of the faces of $M^3_s$ is zero and therefore none of the auxiliary faces inside the cube appears.

Putting $M = M(x, y, z) = M^{(1)}$ we see that

$$C(x, y, z) = \sum_{i=1}^{6} [L; M^{(i)}] = \sum_{\sigma \in S_3(x, y, z)} [L; M(\sigma(x), \sigma(y), \sigma(z))],$$

where $S_3(x, y, z)$ is the group of permutations of $x$, $y$ and $z$. Hence it is enough to calculate $a_3([L; M])$. Setting $(L_iM_j) = (L_iM_0, ..., L_iM_3)$ we have

$$a_3(C(x, y, z)) = \frac{1}{6} \sum_{\sigma \in S_3(x, y, z)} \sum_{i, j = 0}^{3} (-1)^{i+j} \sigma[r_3(L_iM_j) - 2\mu_3(L_iM_j)].$$

Notice that for $x_1, ..., x_6$ in $\mathbb{P}^2$ if two points coincide or four points lie on a line then $r_3(x_1, ..., x_6) = 0$. Hence all the $r_3$-terms are zero except

$$r_3(L_1M_1) = r_3 \begin{bmatrix} 1 & y-z & x-y \\ I & 0 & z-1 & 1-x \\ -1 & 1-y & 0 \end{bmatrix} = r_3 \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & y-z \\ 0 & 1 & x-1 & 0 & 0 & 1 \end{bmatrix},$$

$$r_3(L_3M_0) = r_3 \begin{bmatrix} x & y-z & x-y \\ I & -1 & 0 & y-1 \end{bmatrix} = r_3 \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1-y \\ 0 & 1 & 1-z & 0 & 0 & x-1 \end{bmatrix}.$$

In the last expression of both of the above configurations the first three points lie on a line. By Lemma 3.2, we see that

$$r_3(L_1M_1) = 2\mu_3 \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ (y-z)(x-1) & x-1 & x-y \end{bmatrix},$$

$$r_3(L_3M_0) = 2\mu_3 \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ (x-y)(1-z) & y(1-z) & y-z \end{bmatrix}. (6)$$

Let’s turn to $\mu_3$-terms. There are six non-zero $\mu_3$ terms,

$$\mu_3(L_1M_2) = \mu_3 \begin{bmatrix} y-z \\ z-1 \\ 1-y \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} = \mu_3 \begin{bmatrix} 1 & 0 & y-z & 1 & 1 & x-y \\ 0 & 1 & 1-z & 1 & 0 & x-1 \end{bmatrix}. (7)$$
\[
\mu_3(L_3M_3) = \mu_3 \begin{bmatrix}
 x - y & 1 & x & y - z \\
 y - 1 & 0 & -1 & 0 \\
 1 - x & 0 & z - 1 & 0
\end{bmatrix}
\]
\[
= \mu_3 \begin{bmatrix}
 1 & x - y & 0 & 1 & y & y - z \\
 0 & x - 1 & 1 & 0 & 1 & 1 - z
\end{bmatrix},
\]
and (by Lemma 3.6)
\[
\mu_3(L_0M_0) = \mu_3(L_0M_1) = \mu_3(L_1M_1) = \mu_3(L_3M_0) = \{1\}_3.
\]
Noticing that the last four \(\mu_3\)-terms cancel out each other by taking appropriate signs into account and
\[
2\mu_3(L_1M_3) = \sigma_{\{x, z\}} r_3(L_1M_1), \quad 2\mu_3(L_3M_3) = \sigma_{\{x, y\}} r_3(L_3M_0)
\]
we see that
\[
s_3(C(x, y, z)) = \frac{1}{4} \sum_{s \in S_{\{x, y, z\}}} \sigma[r_3(L_1M_1) - r_3(L_3M_0)] = 0
\]
by applying Lemma 3.5 to (6) and (7).

We now turn to the prism \(P(x, y, z)\) (see Fig. 7). Without loss of generality, we may assume that \(x + y \neq 1\) and \(x + xy \neq 1\) otherwise, using a suitable plane parallel to \(L_3\) we can cut \(P(x, y)\) into two prisms each of which satisfies the above conditions. The coordinates of the vertices of \(P(x, y, z)\) in \(\mathbb{P}^3\) are \(A = [1, 1, 0, 1]\), \(B = [1, 1, y, 1]\), \(C = [1, 1 - y, y, 1]\), \(D = [1, y, y, x]\), \(E = [1, 1, 0, x]\), and \(F = [1, 1, y, x]\). In the picture

![Diagram of a prism](image-url)
we also show how we cut \( P(x, y) \) into three simplices \( M = ABCD, N = ABDE, \) and \( Q = BFDE. \) We choose their orientations as

\[
M = (ACD, ABD, BCD, ABC), \quad N = (ABD, BDE, ABE, ADE),
\]

\[
Q = (BDE, BFE, DEF, BDF),
\]

where the equations of the faces \( ACD, ABD, BCD, ABC, BDE, ABE, \) and \( DEF \) are given by the columns of

\[
\begin{bmatrix}
1 & 1 & -x & y & 1 & 1 & -x & xy & 1 & x \\
-1 & x & 1 & 0 & 0 & x & x & -1 & -1 & 0 \\
-1 & 0 & -1 & 0 & x & 1 & 0 & 0 & 0 & \frac{x}{1} \\
0 & y & 0 & -1 & y & 0 & -1 & & & \\
\end{bmatrix} = 0,
\]

respectively. By definition

\[
a_3(P(x, y)) = \frac{1}{2} \sum_{i,j=0}^{3} (-1)^i+j/(r_3 - 2\mu_3)[\{L_i M_j + (L_i N_j + (L_i Q_j)\}]. \quad (8)
\]

We first consider \( \mu_3 \)-terms. The only non-trivial \( \mu_3 \)-terms are \( \mu_3(L_0 Q_0) = \mu_3(L_0 N_1), \mu_3(L_1 Q_0) = \mu_3(L_1 N_1), \mu_3(L_3 Q_0) = \mu_3(L_3 N_1) \) and

\[
\begin{align*}
\mu_3(L_3 M_0) &= \mu_3 \begin{bmatrix} 1 & 1 & 0 & y & y & 1 \\ 0 & -1 & 1 & x & x & 0 \end{bmatrix}, \\
\mu_3(L_3 N_3) &= \mu_3 \begin{bmatrix} 1 & 0 & 1 & -x & -y & 1 & \end{bmatrix}.
\end{align*}
\]

By using Lemma 3.6 we find that the remaining twelve nonzero \( \mu_3 \)-terms are equal to \( \pm 1 \) and miraculously cancel out each other:

\[
\begin{align*}
\mu_3(L_0 M_2) &= \mu_3(L_0 Q_1) = \mu_3(L_0 Q_3) = \mu_3(L_1 Q_1) = -\{1\}_3, \\
\mu_3(L_0 M_3) &= \mu_3(L_2 M_2) = \mu_3(L_3 M_3) = \mu_3(L_0 N_2) = \{1\}_3, \\
\mu_3(L_1 N_2) &= \mu_3(L_0 Q_2) = \mu_3(L_2 Q_2) = \mu_3(L_3 Q_2) = \{1\}_3.
\end{align*}
\]

Next we consider the \( r_3 \)-terms in (8). All the \( r_2 \)-terms are zero except

\[
\begin{align*}
r_3(L_2 M_2) &= r_3(L_3 M_3) + r_3(L_0 N_2) - r_3(L_1 N_2) + r_3(L_2 N_2) - r_3(L_2 N_2) \\
&- r_3(L_3 N_0) - r_3(L_3 N_2) - r_3(L_1 Q_1) - r_3(L_2 Q_2) - r_3(L_3 Q_2).
\end{align*}
\]

Clearly \( r_3(L_2 N_2) = r_3(L_2 N_3) \) and \( r_3(L_2 Q_3) = -r_3(L_3 N_0) \) by easy computation so they cancel each other. As in the case of cubes we can calculate
the seven remaining terms by Lemma 3.2 and find that \( r_3(L_2 M_2) \) cancels with \( 2 \mu_3(L_1 N_3) \), \( r_3(L_0 N_2) = -6 \{ 1 \}_3 \) and

\[
\begin{align*}
  r_3(L_3 M_3) &= 2 \mu_3 \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & y & 1-x \\ \end{bmatrix}, \\
  r_3(L_1 N_2) &= -2 \mu_3 \begin{bmatrix} 1 & 0 & 1 & 1 & y & 1-x-xy \\ 0 & 1 & 1 & 0 & 1-x \\ \end{bmatrix}, \\
  r_3(L_3 N_2) &= -2 \mu_3 \begin{bmatrix} 1 & 0 & 1 & 1 & 1-x & 1-x-xy \\ 0 & 1 & 1 & 0 & 1-x-y & 1-x-y \\ \end{bmatrix}, \\
  r_3(L_1 Q_1) &= 2 \mu_3 \begin{bmatrix} 1 & 0 & 1 & 1 & 1-x-y & 1-x-xy \\ 0 & 1 & 1 & 0 & -xy & -xy \\ \end{bmatrix}, \\
  r_3(L_3 Q_2) &= 2 \mu_3 \begin{bmatrix} 1 & 0 & 1 & 1 & 1-x-xy & 1-x-y \\ 0 & 1 & 1 & 0 & 1-x & 1-x \\ \end{bmatrix}.
\end{align*}
\]

Applying Lemma 3.5 to simplify the above and combining with the only survivor of \( \mu_3 \)-terms, \( 2 \mu_3(L_3 M_6) \), we see that

\[
a_3(P(x, y)) = -KS \left( \frac{1-x}{1-x-y} \cdot \frac{x}{x-1} \right)
\]

by using the Relation (ii) of \( R_3(F) \)

\[
\{ t \}_3 + \{ 1-t \}_3 + \{ 1-t^{-1} \}_3 = \{ 1 \}_3
\]

with \( t \) equal to

\[
\frac{x}{1-y} - \frac{xy^2}{1-x-y}, \frac{-(1-x)^2}{x(1-x-y)}, \frac{y}{1-x}, \frac{-xy}{1-x}, \frac{1-x}{xy}, \frac{1-x-y}{y(x-1)}.
\]

This concludes our proof of \( a_3(\Pi_3) = 0 \).

3.3. The Maps \( l_n : B_n \to A_n/\Pi_n, n = 2, 3 \)

For \( n \geq 2 \) we define \( \hat{l}_n : \mathbb{Z}[\mathcal{P}_n^+] \to A_n \) as follows: \( \hat{l}_n(\{ 0 \}) = \hat{l}_n(\{ \infty \}) = 0 \) and for \( x \in F^* \)

\[
\hat{l}_n(\{ x \}) = \begin{cases} 
(\text{identified with } [\infty, 0; 1, x] \in A_1 \text{ by cross ratio}) & \text{if } n = 1 \\
([ -1 ]^n A_n(x) & \text{if } n \geq 2.
\end{cases}
\]

Here \( A_n(x) \) corresponds to the classical polylogarithm \( Li_n(x) \) defined at the end of Section 1.
FIG. 8. Some relations of $A_2$. 

**Theorem 3.9.** Modulo prisms $\lambda_n(R_n(F)) = 0$ for $1 \leq n \leq 3$.

**Proof.** The case $n = 1$ is trivial so we assume $n \geq 2$. Then the theorem follows from Proposition 3.11 and Proposition 3.15. 

**Lemma 3.10.** We have $12 A_2(1) = 0$ and

$$A_2(t) + A_2(1 - t) = A_2(1) + \mu[t \otimes (1 - t)]$$

(9)

$$A_2(t) + A_2(t^{-1}) = \frac{1}{2} \mu[t \otimes t] - A_2(1), \quad \forall t \neq 1.$$ 

(10)

**Proof.** By [2, (3.9.1)] we have $12 A_2(1) = 0$. Equation (9) is proved in [3, 5.7] and follows easily from the definition (see the left picture of Fig. 8). Now in the right picture of Fig. 8 we can take $L_3 = \{t_2 = t_0\}$ and use additivity on $L$ to get

$$A_2(x) = [(L_0, L_1, L_2); M] = [(L_0, L_1, L_3); M] + [(L_1, L_2, L_3); M]$$

$$= \frac{1}{2} \mu[(1 - x) \otimes (1 - x)] - A_2(x) \otimes (1 - x)).$$

Changing $x$ to $1 - x$ we see that

$$A_2(1 - x) = \frac{1}{2} \mu[x \otimes x] - A_2(1, 0, \infty, 1 - x)).$$

Adding the last two equalities together, using Eq. (9) we see that

$$A_2(1) + \mu[x \otimes (1 - x)] = \frac{1}{2} \mu[(1 - x) \otimes (1 - x)]$$

$$+ \frac{1}{2} \mu[x \otimes x] - A_2 \left( \frac{x}{x - 1} \right) - A_2 \left( \frac{x - 1}{x} \right).$$

Thus substituting $t$ for $x/(x - 1)$ we get (10). Notice that (10) is valid only for $t \neq 1$ because $x/(x - 1) = 1$ only for $x = \infty$ in which case $(L; M)$ is not admissible anymore. 

[Image of Fig. 8 showing some relations of $A_2$.]
Proposition 3.11. For any five points $x_0, ..., x_4 \in \mathbb{P}^1_\mathbb{F}$,
$$\sum_{i=0}^{4} \lambda_i(r(x_0, ..., x_i, ..., x_4)) = 0$$
modulo squares.

Remark 3.12. The following proof is outlined in [3]. We give the
details here for two reasons: first, the proof uses the above lemma which
is not given explicitly in [3]; second, the proof provides us a hint on
how things may go in the higher case, especially when $n = 3$. See
Proposition 3.15.

Proof. Because of the above lemma we only need to consider the case
where the five points are all distinct. By projective invariance, all such five
term relations can be expressed by the five term relations for the five
distinct points $\infty, x, 1, 0, y \in \mathbb{P}^1_\mathbb{F}$. We want to show that
\[ A_2(r(x, 1, 0, y)) - A_2(r(\infty, 1, 0, y)) + A_2(r(\infty, x, 0, y)) 
- A_2(r(\infty, x, 1, y)) + A_2(r(\infty, x, 1, 0)) = 0 \tag{11} \]
modulo prisms and torsions.

We now consider the pair $A_2(y) = [L; M] \in A_2$ where $L$ is the standard
simplex and $M$ is given by
$$M_0 : t_1 + t_2 = t_0, \quad M_1 : t_1 = t_0, \quad M_2 : t_2 = yt_0.$$ 
Now we can calculate $[L; M]$ in another way. Take $L_3 = \{t_2 = xt_0\}$ (see
Fig. 9). By additivity
$$A_2(r(\infty, 0, 1, y)) = [L; M] = [(L_0, L_1, L_3); M] + [(L_1, L_2, L_3); M].$$
Taking $M_3 = \{t_1 = (1 - x)t_0\}$ and omitting the square $(M_1M_2M_3L_2)$
\[ [(L_0, L_1, L_3); M] = [(L_0, L_1, L_3); (M_0, M_3, M_2)] 
- [(L_0, L_1, L_3); (M_0, M_3, L_2)] 
= A_2(r(\infty, x, 1, y)) - A_2(r(\infty, x, 1, 0)). \]
Taking $M_4 = \{xt_1 + t_2 = xt_0\}$ we get
\[ [(L_1, L_2, L_3); M] = [(L_1, L_2, L_3); (M_0, M_2, M_4)] 
- [(L_1, L_2, L_3); (M_0, M_2, L_4)] 
= -A_2(r(x, 0, \infty, y)) + A_2(r(x, 0, 1, y)). \]
Thus

\[ A_2(r(\infty, 0, 1, y)) = \mu \left[ 1 - \frac{y}{x} \right] \otimes (1 - x) + A_2(r(\infty, x, 1, y)) - A_2(r(\infty, x, 1, 0)) - A_2(r(x, 0, \infty, y)) + A_2(r(x, 0, 1, y)). \]

So Eq. (11) follows from Lemma 3.10.

Before giving the next proposition for \( n = 3 \) we recall that one of the key results in [14] is that \( L_3(z) \) satisfies a 22-term functional equation with three variables given by

\[ R_3(a, b, c) = \{-abc\} + \sum_{\text{cycle}} \left\{ \{ ca-a+1 \} + \{ \frac{ca-a+1}{ca} \} + \{ c \} - \{ 1 \} \right\} 
+ \left\{ \frac{a(bc-c+1)}{ca-a+1} \right\} + \left\{ \frac{bc-c+1}{b(ca-a+1)} \right\} 
- \left\{ \frac{ca-a+1}{c} \right\} \}
\]

where \( \sum_{\text{cycle}} f(a, b, c) = f(a, b, c) + f(b, c, a) + f(c, a, b) \). (There is a misprint in formula (1.16) on [14, p. 208; 15, p. 66] for \( R_3(a, b, c) \). The term \( \{ (bc-c+1) a/(ca-a+1) \} \) should read \( \{ (bc-c+1) a/(ca-a+1) \} \).)

In fact it is essentially the \( (r_3-) \) seven-term relation applied to the following seven points in \( \mathbb{P}^2 \):

\[
[l_0, ..., l_6] = \begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 0 & c \\
0 & 1 & 0 & 1 & a & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & b & 1
\end{bmatrix},
\]
where the columns are the coordinates of the points. Using the notation in Definition 2.6 we have $R_3(a, b, c) = R_3(l_0, ..., l_6) + 3 \{1\}$.

**Lemma 3.13.** In $B_3(F)$, modulo KS relations one has

$$
\sum_{i=0}^{6} r_3(l_0, ..., \widehat{l_i}, ..., l_6) = 6R_3(a, b, c).
$$

**Proof.** Let $s(a, b, c) = r_3(l_1, ..., l_6)$ and $t(a, b) = r_3(l_0, ..., l_5)$. Then one quickly checks that

$$
\sum_{i=0}^{6} r_3(l_0, ..., \widehat{l_i}, ..., l_6) = -r_3(l_0, l_1, l_2, l_4, l_5, l_6) + \bigoplus_{\text{cycle}} [s(a, b, c) + t(a, b)].
$$

We have

$$
\begin{align*}
    r_3(l_0, l_1, l_2, l_4, l_5, l_6) &= r_3 \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & c \\ 0 & 1 & 0 & a & 1 & 0 \\ 0 & 0 & 1 & 0 & b & 1 \end{bmatrix} \\
    &= r_3 \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & abc \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} = -6 \{ -abc \}_3
\end{align*}
$$

by Corollary 3.7. Note that $t(a, b)$ is given by Eqs. (4) and (5) so only $s(a, b, c)$ needs to be displayed explicitly. Using key Lemma 3.2 we have

$$
\begin{align*}
    s(a, b, c) &= r_3 \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & c \\ 1 & 0 & 1 & a & 1 & 0 \\ 0 & 1 & 1 & 0 & b & 1 \end{bmatrix} = r_3 \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & c \\ 1 & 0 & 1 & a & 1 & 0 \\ 0 & 1 & 1 & 0 & b & 1 \end{bmatrix} \\
    &= 2\mu_3 \begin{bmatrix} 1 & 1 & c-a & c & 1-a \\ 0 & 1 & b & c-1 & 1 \end{bmatrix} \\
    &= 2\mu_3 \begin{bmatrix} 1 & 0 & 1 & -abc & bc-bab \\ 0 & 1 & 1 & c-1 & 1 \end{bmatrix}
\end{align*}
$$
\[\begin{align*}
&= 2 \left( \{ ca-a+1 \} - \left\{ \frac{ca-a+1}{1-a} \right\} + \left\{ \frac{ca-a+1}{1-a}(1-a) \right\} \right) \\
&\quad - \left\{ \frac{ca-a+1}{ca-a} \right\} + \left\{ \frac{ca-a+1}{c(ab-b+1)} \right\} - \left\{ \frac{ca-a+1}{1+abc} \right\} \\
&\quad + \left\{ \frac{ca-a+1}{1-a} \right\} - \left\{ \frac{ca-a+1}{ca(ab-b+1)} \right\} + \left\{ \frac{ca-a+1}{ca} \right\} \\
&\quad - \left\{ \frac{ca-a+1}{1-a}(1+abc) \right\} - \left\{ \frac{ca-a+1}{(1-a)(bc-c+1)} \right\} \\
&\quad + \left\{ \frac{bc(ca-a+1)}{(1-c)(1+abc)} \right\} + \left\{ \frac{b(ca-a+1)}{1+abc} \right\} + \left\{ \frac{b(ca-a+1)}{bc-c+1} \right\} \\
&\quad - \left\{ \frac{bc(ca-a+1)}{bc-c+1} \right\} + \left\{ \frac{-b(ca-a+1)}{ab-b+1} \right\} - \left\{ \frac{-b(ca-a+1)}{(1-c)(ab-b+1)} \right\}.
\end{align*}\]

(We intentionally dropped the subscript 3 in the above.)

Now putting everything above together and using
\[\{ t \}_3 + \{ 1-t \}_3 + \{ 1-t^{-1} \}_3 = \{ 1 \}_3\]
for \( t \) equal to all the terms in \( R_3(a, b, c) \) except \( \{ -abc \}_3 \) we get the desired result.

**Corollary 3.14.** Modulo torsions \( R_3(F) \) is generated by \( \{ x \} + \{ 1-x \} + \{ 1-x^{-1} \} - \{ 1 \} \) and \( R_3(a, b, c) \) for \( x, a, b, c \in F \).

**Proof.** From Theorem 2.13 we see that \( R_3(F) \cong \tilde{R}_3(F) \) is generated by \( \{ x \} + \{ 1-x \} + \{ 1-x^{-1} \} - \{ 1 \} \) and \( R_3(l_0, ..., l_6) + 3\{ 1 \} = R_3(a, b, c) \).

Using this corollary we may prove

**Proposition 3.15.** Modulo prisms \( \hat{\lambda}_3(R_3) = 0 \). In fact, for any \( x, y, a, b \) and \( c \) in \( F \)
\begin{align*}
\hat{\lambda}_3(x) + \hat{\lambda}_3(1-x) + \hat{\lambda}_3(1-x^{-1}) &= \hat{\lambda}_3(1), \\
\hat{\lambda}_3(KS(x, y)) &= 0, \\
\hat{\lambda}_3(R_3(a, b, c)) &= 0.
\end{align*}

modulo prisms.

The proof of this proposition is one of the most important steps of this paper and will be given after we list some fundamental formulas relating all
multiple polylogarithmic pairs of weight three to the trilogarithmic pairs in
the next section.
Assuming the two propositions in this section, namely $n_{\nu(n=2,3)}$ send
the relations in $R_{\nu}$ to zero, we see that $n_{\nu}$ induce the maps
$I_{\nu}: B_{\nu} \to A_{\nu}/\Pi_{\nu}$.

3.4. Multiple Polylogarithmic Pairs of Weight Three

The results of this section will be used to prove that $l_3$ is well-defined and
surjective.
For any coplanar points $p_1, ..., p_m$ not on a line we use $\Delta(p_1 \cdots p_m)$
to denote the plane through them.

**Definition 3.16.** Let $(L; M)$ be an admissible pair of simplices in $\mathbb{P}^3_{\nu}$.
We write $M = (ABCD)$ to mean that $A, B, C$ and $D$ are the vertices of $M$
faocing $M_0, M_1, M_2$ and $M_3$ respectively. We say $(L; M)$ is of type $I$
if both of the following two conditions are satisfied:

1. there are two non-coplanar edges $AD$ and $BC$ of $M$ such that $AD$
   includes an $L$-vertex, say $l_3$;

2. the two faces intersecting at $BC$ include a vertex of $L$ (say $l_2$) and
   an edge (say $l_0l_1$) of $L$, respectively (i.e., $l_2 \in \Delta(ABC)$ and
   $l_0l_1 \in \Delta(DBC)$).

The inclusion $C \in L_1$ may or may not be true.
Let $\Gamma(a, b) = [L; M] (b(a-1) \neq 0)$ be of type $I$ where $L$ is the standard
simplex in $\mathbb{P}^3_{\nu}$ and $M = (ABCD)$ is given by

$$
A = [1, 1-a, a, b], \quad B = [1, 1-a, b, b], \\
C = [1, 1-b, b, b], \quad D = [1, 1-a, a, a].
$$

Note that $\Gamma(1, b)$ and $\Gamma(1, 0)$ are not admissible. Also note that $\Gamma(a, a)$
is degenerate and $\Gamma(0, b) = A_3(b)$.

**Lemma 3.17.** If $(L; M)$ is of type $I$ then $(L; M) \in A_3$ modulo prisms. In
fact, for $b \neq 0$ and $a \neq 1$

$$
\Gamma(a, b) = A_3(b) - A_3(a) + \mu \left[ A_2(a) \otimes \frac{a}{b} \right] - \frac{1}{2} \mu \left[ (1-a) \otimes \frac{a}{b} \otimes \frac{a}{b} \right].
$$

**Proof.** Suppose $(L; M)$ satisfies the conditions in the definition. By
cutting $M$ by the plane $\Delta(ADl_2)$ we may assume further that $l_2 \in AB$
and \([L; M] = \Gamma(a, b)\) for suitable choices of \(a\) and \(b\). Let \(E\) and \(F\) be the intersection of \(L_2\) with \(AC\) and \(CD\), respectively. Let \(G\), \(H\), and \(I\) be the intersection of \(l_1l_3EF\) with \(l_1A\), \(l_1D\) and \(BC\), respectively. Let \(J = l_2H \cap L_2\). Geometrically there are three possibilities according to the position of \(F\) relative to points \(C\) and \(D\) (see Fig. 10). In the left picture we see that

\[
M = (FEIC) - (FJHD) - \text{Prism}(GHIADB) - \text{Prism}(AEGDHJ).
\]

So in this case the lemma follows from the fact that \([L; (FEIC)] = A_3(b)\) and \([L; (FJHD)] = A_3(a)\). The other two cases can be treated similarly. \(\blacksquare\)

**Remark 3.18.** Type \(\Gamma\) is named after A. Goncharov because it was him who first predicted the existence of relations such as

\[
Li_{1,2}(x, y) = \text{Li}_1^\natural(x) - \text{Li}_1^\natural\left(\frac{x-xy}{1-xy}\right) + \text{Li}_1^\natural(xy) - L_3\left(\frac{y-xy}{1-xy}\right)
\]

\[
+ Li_3(y) - Li_3(xy) - \log(1-xy)(Li_2(x) + Li_2(y))
\]

\[
- \frac{1}{2} \log^2\left(\frac{1-x}{1-xy}\right) \left(\frac{1-y}{1-xy}\right).
\]

where

\[
\text{Li}_1^\natural(x) = Li_1(1) - Li_3(1-x) + Li_2(1) \log(1-x) - \frac{1}{2} \log(x) \log(1-x).
\]

This particular identity was discovered by Zagier. Using the identities \(Li_{1,1}(y, x) = -Li_2(y) \log(1-x) - Li_{1,2}(x, y) - Li_3(xy)\) and \(Li_2(x) + Li_2(1-x) + \log(1-x) \log(x) = Li_2(1)\) we get
$L_{2,1}(y, x) = L_3(1 - xy) - L_3(1) - \log(1 - xy) L_2(1)$

$$+ L_3(1 - x) - L_3 \left( \frac{1 - x}{1 - xy} \right) - \log(1 - xy) L_2(1 - x)$$

$$+ L_3 \left( \frac{y - xy}{1 - xy} \right) - L_3(y) - \log \left( \frac{1 - x}{1 - xy} \right) L_2(y)$$

$$+ \frac{1}{2} \log(y) \log^2(1 - xy).$$

The first three lines of the above expression each corresponds to a pair of simplices of type $\Gamma$ modulo prisms. Therefore it is the analytic version of the next lemma. Every other result of geometric nature in this section corresponds to some analytic version. We leave the checking of this to the interested readers. The rule of the correspondence is

$$(-1)^{l_1} A_{m_1, \ldots, m_l}(x_1, \ldots, x_l) \leftrightarrow L_{m_1, \ldots, m_l}(x_1, \ldots, x_l).$$

**Lemma 3.19.** *Modulo prisms $A_{2,1} < A_3$. In fact, for $x \neq 1, xy \neq 1$*

$$A_{2,1}(y, x) = A_3(1) - A_3(1 - xy) - A_3(1 - x)$$

$$+ A_3 \left( \frac{1 - x}{1 - xy} \right) + A_3(y) - A_3 \left( \frac{y - xy}{1 - xy} \right)$$

modulo prisms, and for $x \neq 1, y \neq 0$

$$A_{2,1}(y, x) = A_3(1) - A_3(1 - xy) - A_3(1 - x)$$

$$+ A_3 \left( \frac{1 - xy}{1 - x} \right) + A_3 \left( \frac{1}{y} \right) - A_3 \left( \frac{1 - xy}{y - xy} \right)$$

modulo prisms.

**Proof.** Let $A_{2,1}(y, x) = -[L; M]$ where $L$ is the standard simplex and $M = (ABCD)$ is given by

$$A = [1, 0, 1, 1], \quad B = [1, 0, 1, 1 - x],$$

$$C = [1, 1, 1, 1 - x], \quad D = [1, 1, 1 - xy, 1 - x].$$
Here, for the sake of convenience we exchange $L_1$ and $L_2$ so that a minus sign appears. We notice that if $x = 1$ then $(L; M)$ is not admissible.

Step (1). Form the prism $(AEFBCD)$ where $E = [1, 1, 1, 1]$ and $F = [1, 1, 1 - xy, 1]$. Let $G = [1, 1, 1 - x, 1 - x]$ and $H = [1, 1, 1 - xy, 1 - xy]$ be the intersection of $\Delta(l_0l_1AE)$ with $DC$ and $FD$, respectively (see the left picture in Fig. 11). Then we have

$$M = (ABCD) = \text{Prism}(AEFBCD) - (HFEA) - (AECG) + (AHGD),$$

where $[L; (HFEA)] = \Gamma(1 - xy, 1)$ and $[L; (AECG)] = \Gamma(1 - x, 1)$ are both of type $\Gamma$ and by Lemma 3.17

$$[L; (HFEA)] = A_3(1) - A_3(1 - xy), \quad [L; (AECG)] = A_3(1) - A_3(1 - x)$$

modulo prisms. We now turn to $(AHGD)$.

Step (2). Let $I = AH \cap l_0l_1 = (xy, 1, 0, 0)$ (note that $l_0l_1 \in \Delta(AGH)$). Cut $L$ into $L' = (l_0l_1l_2)$ and $L'' = (l_1l_2l_3)$ (see the right picture in Fig. 11). Then both $[L'; (AHGD)]$ and $[L''; (AHGD)]$ are of type $\Gamma$. In fact, for both $L'$ and $L''$ we can take $GD$ and $HA$ as the two special edges of $M$ because the special $L$-edge included in $\Delta(HAG)$ is $l_0l_1l_1l_1$. Take

$$g' = \begin{pmatrix} 1 & -xy & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \text{PGL}_3(F), \quad \text{and} \quad g'' = \begin{pmatrix} 1 & -xy & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \text{PGL}_3(F)$$

and use projective invariance and Lemma 3.17 we get

$$[L'; (AHGD)] = [L; g'(AHGD)] = \Gamma\left( \frac{1 - x}{1 - xy}, 1 \right) = A_3(1) - A_3\left( \frac{1 - x}{1 - xy} \right)$$

FIG. 11. $A_3(y, x)$ decomposed into trilogarithmic pairs.
and (note that $g"L" = (l_1, l_0, l_2, l_3)\))
\[ L'; (AHGD) ] - [ L; g"(AHGD) ] \]
= \[-F\left( \frac{y - xy}{1 - xy}, y \right) = A_3\left( \frac{y - xy}{1 - xy} \right) - A_3(y) \].

Put steps (1) and (2) together we get Eq. (15).

Now if in the second step we take $I = AG \cap l_0 l_1$ and repeat its argument we arrive at (16). This finishes the proof of the lemma.

Remark 3.20. The pictures in the above proof are valid only when $F = C$ and $M$ is a real simplex with $0 \leq x, y \leq 1$, but the proof itself can be carried through easily for all other fields. This is true for all the proofs in the rest of this section.

Lemma 3.21. Modulo prisms $A_{1,2} \subset A_3$. Moreover,

\[ A_{1,2}(1, 1) = -A_3(1), \quad A_{1,2}(1, y) = -A_3(y) - A_3\left( \frac{y}{y - 1} \right) \quad \forall y \neq 1 \]  \hspace{1cm} (17)

modulo prisms. If $x \neq 1, xy \neq 1$ then

\[ A_{1,2}(x, y) = A_3\left( \frac{1}{1 - x} \right) - A_3\left( \frac{1 - xy}{1 - x} \right) - A_3(y) + A_3\left( \frac{y - xy}{1 - xy} \right) - A_3\left( \frac{xy}{xy - 1} \right) \]  \hspace{1cm} (18)

modulo prisms. If $y \neq 1$ then

\[ A_{1,2}(x, y) = A_3(xy) + A_3\left( \frac{y}{y - 1} \right) - A_3\left( \frac{y - xy}{y - 1} \right) + A_2(1, x, \frac{y}{y - 1}) - A_2(1, \frac{y}{y - 1}) \]  \hspace{1cm} (19)

Proof. Let $L$ be the standard simplex and $M = (ABCD)$ where

\[ A = [1, 1 - y, 1, 1 - xy], \quad B = [1, 1 - y, 1, 1], \]
\[ C = [1, 1, 1, 1], \quad D = [1, 1, 0, 1]. \]

Then $[L; M] = A_{1,2}(x, y)$. After exchanging $L_1$ and $L_2$ we see that by definition $A_{1,2}(1, 1) = -A_3(1)$.
First let us assume $xy \neq 1$. We construct a prism $(AEFBCD)$ where $E = [1, 1, 1, 1-xy]$ and $F = [1, 1, 0, 1-xy]$. Let $G = [1, 1, 1/(1-xy), 1]$, and $H = [1, 1, 0, 0]$ be the intersections of $\triangle(AI_0I_1)$ with $CD$ and $DF$, respectively. Then we have (see Fig. 12)

$$M = \text{Prism}(AEFBCD) - (AEFG) - (AEFH) + (AHGD).$$

Clearly $[L; (AEFH)] = A_3(y)$ while $[L; (AEFG)]$ is of type $I$ and by Lemma 3.17

$$[L; (AEFG)] = \begin{cases} 
\frac{1}{6} \mu[(1-y) \otimes (1-y) \otimes (1-y)] & \text{if } x = 1 \\
A_3 \left( \frac{1-xy}{1-x} \right) - A_3 \left( \frac{1}{1-x} \right) & \text{if } x \neq 1
\end{cases}$$

modulo prisms. We now turn to $(AHGD)$. Let $I = [1, 1-x^{-1}, 0, 0] = AG \cap I_0I_1$. Cutting $L$ into $L' = (I_0I_2I_3)$ and $L'' = (II_1I_2I_3)$ we have

$$[L; (AHGD)] = [L'; (AHGD)] + [L''; (AHGD)] = A_3 \left( \frac{x-xy}{1-xy} \right) - A_3 \left( \frac{xy}{xy-1} \right)$$

by projective invariance. Indeed, one may use

$$g = \begin{pmatrix} 1 & x & 0 & 0 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \text{PGL}_3(F), \quad \text{and} \quad g^* = \begin{pmatrix} x & 0 & 0 & 0 \\ x-1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \text{PGL}_3(F).$$
respectively, to transform $L'$ and $L''$ to $L$ and note that $g''([L''; (AHGD)]) = -[L; g'(AHGD)]$. Putting everything together we get (17) and (18).

To derive (19) we consider another construction of $A_{1,2}(x, y)=[L; M]$ as shown by the left picture in Fig. 13 where $L$ is the standard simplex and $M=(ABCD)$ is given by

$$A = \begin{pmatrix} 1, -\frac{1}{xy}, -\frac{1}{y}, 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1, -\frac{1}{xy}, -\frac{1}{y}, 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1, -\frac{1}{xy}, 1 - \frac{1}{y}, 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1, 1 - \frac{1}{xy}, 1 - \frac{1}{y}, 1 \end{pmatrix}.$$

Take $I = (1, 0, -\frac{1}{y}, 0)$ and cut $L$ into $L'=(l_1l_2l_3)$ and $L''=(l_0l_1l_2)$. Then by additivity

$$[L; M] = [L'; M] + [L''; M].$$

Take

$$g' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \text{PGL}_3(F),$$

and

$$g'' = \begin{pmatrix} 0 & 0 & -y & 0 \\ 0 & -xy & 0 & 0 \\ 1/y & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 - y \end{pmatrix} \in \text{PGL}_3(F),$$

respectively, we see that

$$[L'; M] = A_3(xy), \quad [L''; M] = [g''L''; g''M] = -[L; g''M].$$
where $g^*M = (EFGH)$ is given by

$$E = (1, 1, 0, 0), \quad F = (1, 1, 0, 1 - y),$$
$$G = \left(1, \frac{1}{1 - y}, 1, 1\right), \quad H = \left(1, \frac{1 - xy}{1 - y}, 1, 1\right).$$

See the right picture of Fig. 13. Let $J = GH \cap \triangle(l_2EF) = (1, 1, 1, 1)$ and $K = EF \cap \triangle(l_2GH) = (1, 1, 0, 1)$. Note that $(EFGH)$ is oriented positively while $(EFJH)$ and $(EFJG)$ are oriented negatively and we have

$$(EFGH) = -(EFJH) + (EFJG)$$
$$= -(EKJH) + (FKJH) + (EKJG) - (FKJG).$$

Now by definition

$$(EKJH) = A_3\left(\frac{y}{y - 1}\right), \quad (EKJG) = A_3\left(\frac{y - xy}{y - 1}\right),$$
$$\quad (FKJH) = A_{2, 1}\left(1, \frac{y}{y - 1}\right), \quad (FKJG) = A_{2, 1}\left(1 - x, \frac{y}{y - 1}\right).$$

This finishes the proof of the lemma. □

**Corollary 3.22.** For any $t \neq 0$,

$$A_3(t) + A_3(1 - t) + A_3(1 - t^{-1}) = A_3(1) \quad \text{and} \quad A_3(t) = A_3(t^{-1}) \quad (20)$$

modulo prisms.

Proof. All the calculation in this proof is carried out by modulo prisms.

From Eq. (19) of Lemma 3.21 we get for $y \neq 1$

$$A_{1, 2}(1, y) = A_3(y) + A_3\left(\frac{y}{y - 1}\right) - A_{2, 1}\left(1, \frac{y}{y - 1}\right).$$

Comparing this to (17) of Lemma 3.21 we see that for $y \neq 1$

$$A_{2, 1}\left(1, \frac{y}{y - 1}\right) = 2A_3(y) + 2A_3\left(\frac{y}{y - 1}\right).$$
By Eq. (15) of Lemma 3.19 we have for \( x \neq 1 \)

\[
A_{2,1}(1, x) = 2A_3(1) - 2A_3(1 - x).
\]

Thus for \( y \neq 1 \)

\[
A_3(1) = A_3\left(\frac{1}{1-y}\right) + A_3(y) + A_3\left(\frac{y}{y-1}\right).
\] (21)

Taking \( t = 1/(1 - y) \) we get for \( t \neq 0 \)

\[
A_3(1) = A_3(t) + A_3(1 - t^{-1}) + A_3(1 - t).
\]

From Remark 2.7(1) we get for \( t \neq 0 \)

\[
A_3(t) = A_3\left(\frac{1}{t}\right).
\]

The proof of the corollary is now complete.

**Corollary 3.23.** One has

\[
A_{1,2}(x, y) + A_{2,1}(y, x) = A_3(xy)
\]

modulo prisms.

**Proof.** Add (19) and (15) and use the relations in the last corollary.

**Lemma 3.24** Modulo prisms \( A_{1,1,1} \subset A_3 \). Moreover, if \( z \neq 1 \)

\[
A_{1,1,1}(1, 1, z) = \frac{1}{6} \mu [(1 - z) \otimes (1 - z) \otimes (1 - z)].
\] (22)

If \( x \neq 1 \) and \( z \neq 1 \) then

\[
A_{1,1,1}(x, 1, z) = A_3\left(\frac{x - xz}{x - 1}\right) - A_3\left(\frac{x}{x - 1}\right)
\]

\[- \mu \left[A_3\left(\frac{x}{x - 1}\right) \otimes (1 - z)\right]
\]

\[+ \frac{1}{2} \mu [(1 - x) \otimes (1 - z) \otimes (1 - z)].
\] (23)
If \( y \neq 1 \) and \( z \neq 1 \) then
\[
A_{1,1,1}(1, y, z) = A_3 \left( \frac{1-yz}{1-y} \right) - A_3 \left( \frac{1}{1-y} \right)
\]
\[
- \mu \left[ A_2 \left( \frac{1-yz}{1-y} \right) \otimes (1-yz) \right]
\]
\[
+ \frac{1}{2} \mu \left[ \left( \frac{y-yz}{y-1} \right) \otimes (1-yz) \otimes (1-yz) \right].
\] (24)

If \( x \neq 1, y \neq 1, \) and \( z \neq 1 \) then
\[
A_{1,1,1}(x, y, z) = A_{2,1} \left( \frac{x-xy}{x-1} \cdot \frac{1-yz}{1-y} \right) - A_{2,1} \left( \frac{x-xy}{x-1} \cdot \frac{1}{1-y} \right)
\]
\[
- \mu \left[ A_2 \left( \frac{x}{x-1} \right) \otimes (1-z) \right] + \mu [A_{1,1}(y, z) \otimes (1-x)].
\] (25)

Proof. Let \( A_{1,1,1}(x, y, z) = [L; M] \) where \( L \) is the standard simplex and \( M = (ABCD) \) is given by
\[
A = (1, 1, 1, 1), \quad B = (1, 1, 1 - yz, 1 - z),
\]
\[
C = (1, 1 - xyz, 1 - yz, 1 - z), \quad D = (1, 1, 1 - z).
\]

Clearly \( [L; M] \) is non-admissible if \( z = 1 \).

Equation (22) follows easily by definition so there are three cases left: (1) \( x = 1, y \neq 1, (2) x \neq 1, y \neq 1, \) and (3) \( x \neq 1, y = 1. \)

Let us first assume that \( y \neq 1. \) We use the idea to decompose a pair of type \( \Gamma \) into trilogarithmic pair to decompose \( [L; M] \). Let \( E \) and \( F \) be the intersection of \( CD \) and \( AC \) with \( L_2 \), respectively. Then \( F = (1, 1 - x, 0, 1 - (1/y)). \) Put \( J = EF \cap \Delta(l_1l_2A) \) and let \( G, H, \) and \( I \) be the intersection of plane \( \Delta(l_2EF) \) with \( l_1D, l_1A, \) and \( BC \), respectively (see Fig. 14). Clearly we have
\[
M = \text{Prism}(DBAGIH) - \text{Prism}(DEGAJH) - (FIHJA) + (FEIC).
\]
(Not that both \( (FIHJA) \) and \( (FEIC) \) are negatively oriented.)

Case (1). If \( x = 1 \) then \( [L; M] \) is of type \( \Gamma. \) By exchanging \( L_1 \) and \( L_3 \) and using Lemma 3.17 we get
\[
[L; M] = -\Gamma \left( \frac{1-yz}{1-y}, \frac{1}{1-y} \right)
\]
which is (24).
Case (2). If \( x \neq 1 \) then
\[
(FJHA) = A_{x,1} \left( \frac{x-xy}{x-1}, \frac{1}{1-y} \right), \quad (FEIC) = A_{y,1} \left( \frac{x-xy}{x-1}, \frac{1-yz}{1-y} \right)
\]
which imply (25).

Finally we deal with

Case (3). \( y = 1 \) and \( x \neq 1 \). Then \([L; M]\) is again of type \( \Gamma \). By Lemma 3.17 we see that
\[
[L; M] = \Gamma \left( \frac{x}{x-1}, \frac{x-xy}{x-1} \right)
\]
which is (23).

This completes the proof of the lemma.  

Combining Eq. (25) and Eq. (15) we get

**Corollary 3.25.** If \( x \neq 1 \), \( y \neq 1 \) and \( z \neq 1 \) then modulo prisms

\[
A_{1,1,1}(x, y, z) = A_{1,1,1}(x, x^{-1}, z) = A_{3}(1-x) - A_{3} \left( \frac{1-xyz}{1-x} \right) - A_{3} \left( \frac{y}{y-1} \right) + A_{3}(1-x) + A_{3}(1-z) + A_{3}(xy) - A_{3} \left( \frac{y-xy}{y-1} \right).
\]

In particular,

\[
A_{1,1,1}(x, x^{-1}, z) = 2A_{3}(1-x) - 2A_{3} \left( \frac{1-z}{1-x} \right).
\]
3.5. Proof of Proposition 3.15: $l_3$ Is Well Defined

Because (12) was proved as Corollary 3.22 in Subsection 3.7 we only need to prove Eqs. (13) and (14). We will first derive Eq. (13) although it is a special case of (14), because we will need the former to prove the latter. This is the reason we list Eq. (13) in our proposition.

We are going to use two methods to calculate the trilogarithmic decomposition of the following pair $(L, M)$ of simplices in $A_3$: $L$ is the standard simplex and $M = (ABCD)$ is given by (see Fig. 15)

$$M_0 = \Delta(BCD); \quad M_1 = \Delta(ACD); \quad M_2 = \Delta(ABD); \quad M_3 = \Delta(ABC)$$

which is oriented negatively. Equations (13) and (14) will be proved by comparing these two decompositions.

By simple calculation one has

$$(ABCD) = \begin{pmatrix} 1 + abc & bc - b + 1 & 1 & ca - c + 1 \\ ca & c & 0 & ca \\ 1 & 1 - b & 1 & 1 \\ -c & bc - c & 0 & -c \end{pmatrix},$$

where the columns represent the coordinates of the four points in $\mathbb{P}_3^1$.

1. First Method. Let $E = (1, 0, 1, -c) = l_3 C \cap AB$ (see Fig. 15) and cut $M$ into $M' = (EBCD)$ and $M'' = (AECD)$ which are oriented negatively and positively, respectively. By additivity

$$[L; M] = [L; M'] + [L; M''].$$

![FIG. 15. Decomposition of this pair into trilogarithmic pairs.](image-url)
FIG. 16. Left, (1) cut $M$ into two parts. Right, (2) cut $L$ into two parts.

(1) $[L; M^*]$. Let
\[
h = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/\text{ca} & 0 & 0 \\ 0 & 0 & 0 & -1/c \end{pmatrix} \in \text{PGL}_3(F).
\]

By projective invariance
\[
[L; M^*] = [hL; hM^*] = [L; hM^*],
\]
where
\[
hM^* = h(\text{AECD}) \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 + abc & 1 & 1 & \text{ca} - c + 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}.
\]

Still using $(\text{AECD})$ for $h(\text{AECD})$ we have the left picture of Fig. 16, where $R = (1, 1, 1, 1) = AD \cap l_2E$. Note that $hM^*$ and $(\text{CERA})$ (resp. $(\text{CERD})$) are oriented negatively (resp. positively) we have
\[
[L; h M^*] = [L; (\text{CERA})] - [L; (\text{CERD})] = A_3(-abc) - A_3(c(1-a)).
\]

(2) $[L; M^*]$. Let $I = (1, 1, 0, 0) = l_0l_1 \cap BD$ (see the right picture of Fig. 16) and cut $L$ into $L' = (l_0l_1l_3)$ and $L'' = (l_1l_2l_3)$. By additivity on $L$
\[
[L; M^*] = [L'; M^*] + [L''; M^*].
\]
Let
\[
g' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & (1-b)/c & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1/c \end{pmatrix}
\]
$g'M'$ is equal to a prism and $g'M'$ is decomposed into $A(3)$.

and

$$g' = \begin{pmatrix}
1 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1-c & 0 \\
0 & 0 & 0 & (c-1)/c
\end{pmatrix} \in \text{PLG}_3(F).$$

By projective invariance

$$[L; M'] = [g'L'; g'M'] + [g"L"; g"M"] = -[L; g'M'] - [L; g"M'].$$

(i) $g'M'$. By abuse of notation we have (see the left picture of Fig. 17)

$$g'M' = g'(EBCD) = (EBCD) = \begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 0 & a(1-b) \\
1 & 1-c & 1 & 1-c \\
1 & 1 & 0 & 1
\end{pmatrix}.$$ 

Note that $(EBCD)$, $(CEFB)$, and $(CEGD)$ are oriented negatively while $(CIGD)$ and $(CHFB)$ are oriented positively and we have

$$(EBCD) = - \text{Prism}(BFHDGI) + (CEGD) - (CIGD) - (CEFB) + (CHFB) = 0$$

because $(CEGD)$ cancels with $(CIGD)$ while $(CIGD)$ cancels with $(CHFB)$.

Thus modulo prisms

$$[L; g'M'] = 0.$$
(ii) $g^*M'$. By abuse of notation we have (see the right picture of Fig. 17)

$$g^*M' = g^*(EBCD) = (EBCD)$$

$$\begin{pmatrix}
1 & (1-b)(1-c) & 1 & 1-c \\
1 & bc-b+1 & 1 & ca-c+1 \\
1-c & (1-b)(1-c) & 1 & 1-c \\
1-c & (1-b)(1-c) & 0 & 1-c
\end{pmatrix}$$

$$= \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & bc-b+1 & 1 & ca-c+1 \\
1-c & (1-b)(1-c) & 1 & 1-c \\
1-c & 1 & 0 & 1
\end{pmatrix}.$$ 

Because $(EBCD)$ is positively oriented we have

$$g^*M' = \text{Prism}(QIBCJD) + (CKJD) - (CPJD) - (CKIB) + (CQIB)$$

$$+ \text{Prism}(BIHDJG) - (EKJD) + (EGJD) + (EKIB) - (EHIB).$$

Thus modulo prisms

$$- [L; g^*M'] = A_{1,2} \left( \frac{1}{a}, \frac{ca}{c-1} \right) + A_{2,1} \left( \frac{1}{a}, \frac{ca}{c-1} \right)$$

$$- A_{1,2} \left( 1-b, \frac{-c}{(1-b)(1-c)} \right)$$

$$- A_{2,1} \left( 1-b, \frac{-c}{(1-b)(1-c)} \right) - A_{1,1} \left( \frac{1}{a}, \frac{ca}{c-1} \right)$$

$$- A_{1,1} \left( \frac{1}{a}, \frac{ca}{c-1} \right) + A_{1,1} \left( 1, 1-b, \frac{-c}{(1-b)(1-c)} \right)$$

$$+ A_{1,1} \left( 1-b, \frac{1}{1-b}, \frac{e}{c-1} \right)$$

which can be written as linear combinations of trilogarithmic pairs using results in Subsection 3.4.

Finally, using $A_3(t) + A_3(1-t) + A_3(1-t^{-1}) = A_3(1)$ for

$$t = \frac{ca - c + 1}{ca}, \quad \frac{bc - b + 1}{bc}, \quad \frac{bc - b + 1}{c}$$
we have
\[
\begin{bmatrix}
L & M
\end{bmatrix} = \begin{bmatrix}
L & hM
\end{bmatrix} - \begin{bmatrix}
L & g'M
\end{bmatrix}
\]
\[
= A_3(-abc) + A_3(a) + A_3\left(\frac{bc-b+1}{bc}\right)
\]
\[
+ A_3\left(\frac{ca-c+1}{ca}\right) + A_3(ca-c+1)
\]
\[
+ A_3(bc-b+1) - A_3\left(\frac{bc-b+1}{c}\right)
\]
\[
- A_3\left(\frac{ca-c+1}{a}\right) - A_3(1) - A_3(1-b),
\]
modulo prisms.

(II) Second Method. Let \( S = (0, 1, -b, 0) \in I_1 I_2 \) and cut \( \overline{L} = (l_0 S) \) and \( \overline{L} = (l_0 S) \). By additivity
\[
\begin{bmatrix}
L & M
\end{bmatrix} = \begin{bmatrix}
\overline{L} & M
\end{bmatrix} + \begin{bmatrix}
\overline{L} & M
\end{bmatrix}.
\]
Let
\[
\tilde{g} = \begin{pmatrix}
0 & b & 1 & 0 \\
0 & 1/c & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/c
\end{pmatrix}
\quad \text{and} \quad
\tilde{g} = \begin{pmatrix}
0 & b & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1/c & 0
\end{pmatrix}
\in \text{PGL}_3(F).
\]
By projective invariance
\[
\begin{bmatrix}
L & M
\end{bmatrix} = [\tilde{g} \overline{L}; \tilde{g} M] + [\tilde{g} \overline{L}; \tilde{g} M] = -[\tilde{g} \overline{L}; \tilde{g} M] - [\tilde{g} \overline{L}; \tilde{g} M].
\]
(1) \( [\tilde{L}; \tilde{g} M] \). By abuse of notation
\[
\tilde{g} M = (ABCD) = \begin{pmatrix}
1+abc & bc-b+1 & 1 & 1+abc \\
1+abc & bc-b+1 & ac-c+1 & 1 \\
1 & 1-b & 0 & 1
\end{pmatrix}.
\]
Let \( I = (1, 0, 0, 1/c) = \triangle(BCD) \cap l_0 I_2 \). Cutting \( \overline{L} \) into \( \overline{L} = (l_0 H_1 I_2) \) and \( \overline{L} = (l_0 H_1 I_2) \) and using additivity we get
\[
\begin{bmatrix}
L & \tilde{g} M
\end{bmatrix} = [\tilde{L}'; \tilde{g} M] + [\tilde{L}'; \tilde{g} M].
\]
Put
\[
\tilde{g}' = \begin{pmatrix} 1 & 0 & 0 & -c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{g}^\circ = \begin{pmatrix} 1 & 0 & 0 & -c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \in \text{PGL}_4(F).
\]

Then
\[
[L; \tilde{g} M] = [L; \tilde{g} \tilde{L}]; \tilde{g} \tilde{g} M] + [\tilde{g} \tilde{g} M; \tilde{g} \tilde{g} M] = -[L; \tilde{g} \tilde{g} M] + [L; \tilde{g} \tilde{g} M].
\]

(i) \( \tilde{g} \tilde{g} M \). By abuse of notation
\[
\tilde{g} \tilde{g} M = (ABCD) = \begin{pmatrix} 1 + abc - c & 2bc - b - c + 1 & 1 & 1 + abc - c \\
a & 1 & 0 & a \\
1 + abc & bc - b + 1 & 1 & ac - c + 1 \\
1 + abc & bc - b + 1 & 1 & 1 + abc \end{pmatrix}.
\]

Let \( E = (1 + abc - c, 1 - b, 1 + abc, 1 + abc) = l_1 A \cap BC \). Then \( [L; \tilde{g} \tilde{g} M] = [L; (EABD)] - [L; (EACD)] \) (see the left picture of Fig. 18).

To treat \( [L; (EACD)] \) let \( H = Cl_1 \cap ED \). Let \( F \) and \( G \) be the intersection of the face \( \triangle(El_1) \) with \( Cl_1 \) and \( Hl_1 \) respectively (see the middle picture of Fig. 18). Let \( I \) and \( J \) be the intersection of the face \( \triangle(ACD) \) with \( Cl_1 \) and \( Hl_1 \), respectively.

Note that \( (EACD) \), \( (CHJD) \) and \( (CHGE) \) are oriented positively and \( (CIJA) \) and \( (CFGE) \) are oriented negatively we have by Lemma 3.19
\[
(E\tilde{A}\tilde{C}D) = \text{Prism}(FGEIJA) - (CIJA) + (CHJD) - (CHGE) + (CFGE)
\]
modulo prisms because (CIJA) cancel with (CFGE).

**FIG. 18.** Second method: part (1)(i).
To treat \([L; (EABD)]\) we let \(P = B\bar{I}_3 \cap ED\), \(Q = \bar{l}_1P \cap AD\) and \(R = \bar{l}_2P \cap AE\) (see the right picture of Fig. 18). Then by Lemma 3.24

\[
(E\bar{A}B\bar{D})' = -(BPRE) + (BPRA) - (BPQA) + (BPQD)
\]

\[
= -A_{1,1,1} \left( \begin{array}{ccc}
bc - b + 1 & 1, & -bc^2(ab - a + 1) \\
\frac{c}{bc} & -bc^2(ab - a + 1) & 1 + abc - c(bc - b + 1)
\end{array} \right)
\]

\[
+ A_{1,1,1} \left( \begin{array}{ccc}
1 - c & -bc^2(ab - a + 1) \\
\frac{bc}{c} & 1 + abc - c(bc - b + 1)
\end{array} \right)
\]

\[
+ A_{1,1,1} \left( \begin{array}{ccc}
1 & -bc^2(ab - a + 1) \\
\frac{1 - c}{bc - b + 1} & 1 + abc - c(bc - b + 1)
\end{array} \right)
\]

\[
- A_{1,1,1} \left( \begin{array}{ccc}
1 - c & -bc^2(ab - a + 1) \\
\frac{bc}{bc - b + 1} & 1 + abc - c(bc - b + 1)
\end{array} \right)
\]

By Lemma 3.19 and Lemma 3.24

\[ [L; \tilde{g}'\tilde{g}M] = (E\bar{A}B\bar{D})' - (E\bar{A}\bar{C}D)'
\]

\[
= -A_3(1) + A_3 \left( \frac{1 + abc}{ac - c + 1} \right) + A_3(a - ab)
\]

\[
- A_3 \left( \frac{a(1 + abc)(1 - b)}{ac - c + 1} \right) + A_3(1 - c)
\]

\[
- A_3 \left( \frac{(1 - c)(1 + abc)}{ac - c + 1} \right) + A_3 \left( \frac{-bc(ac - c + 1)}{(1 + abc)(1 - c)(1 - b)} \right)
\]

\[
- A_3 \left( \frac{(1 - c)(1 - b)}{bc} \right)
\]

(ii) \(\tilde{g}'\tilde{g}M\). By abuse of notation

\[
\tilde{g}'\tilde{g}M = (ABCD) = \left( \begin{array}{cccc}
1 + abc & bc - b + c + 1 & 1 + abc & c \\
1 + abc & bc - b + 1 & 1 & ac - c + 1 \\
1 & 1 - b & 0 & 1 \\
a & 1 & 0 & a
\end{array} \right)
\]
Let $E = (1 + abc - c, ac - c + 1, a - ab, a) = l_2 D \cap BC$. Then $[L; g^g M] = [L; (EABD)] - [L; (EACD)]$ (see the left picture of Fig. 19).

To treat $[L; (EACD)]$ let $H = Cl_3 \cap EA$. Let $F$ and $G$ be the intersection of the face $\triangle(El_1 l_3)$ with $Cl_2$ and $Hl_2$, respectively (see the middle picture of Fig. 19). Let $I$ and $J$ be the intersection of the face $\triangle(ADl_2)$ with $Cl_1$ and $Hl_2$, respectively. Because $(EACD), (CIJD),$ and $(CFGE)$ are positively oriented while $(CHJA)$ and $(CHGE)$ are negatively oriented we have

$$(EACD)^* = (CIJD) - (CHJA) - \text{Prism}(FGEIJD) + (CHGE) - (CFGE)$$

$$= -A_3 \left( -\frac{c}{1 + abc - c} \right) + A_3 \left( \frac{abc - ac}{1 + abc - c} \right)$$

modulo prism since $(CIJD)$ cancels with $(CFGE)$.

To treat $[L; (EABD)]$ we let $P = Bl_3 \cap EA$, $Q = l_1 P \cap ED$, and $R = l_2 P \cap AD$ (see the right picture of Fig. 20). Because $(EABD), (BPRA)$, and $(BPQD)$ are negatively oriented while $(BPRA)$ and $(BPQD)$ are positively oriented we have

$$(EABD)^* = -(BPRA) + (BPRA) - (BPQD) + (BPQ)$$

$$= -A_{1,1,1,1} \left( \frac{c - bc}{bc - b + 1} \cdot \frac{-bc}{(1-c)(1-b)} \cdot \frac{(1-c)(ab - a + 1)}{1 + abc - c} \right)$$

$$+ A_{1,1,1,1} \left( \frac{(1-b)^2(1-c)}{-b(bc - b + 1)} \cdot \frac{-bc}{(1-c)(1-b)} \cdot \frac{(1-c)(ab - a + 1)}{1 + abc - c} \right)$$

$$+ A_{1,1,1,1} \left( \frac{-b(bc - b + 1)}{(1-b)^2(1-c)} \cdot \frac{c - bc}{bc - b + 1} \cdot \frac{(1-c)(ab - a + 1)}{1 + abc - c} \right)$$

$$- A_{1,1,1,1} \left( \frac{bc - b + 1}{c - bc} \cdot \frac{c - bc}{bc - b + 1} \cdot \frac{(1-c)(ab - a + 1)}{1 + abc - c} \right).$$
Hence by Lemma 3.24

$$- [L; \breve{g}^* g M] = (E\overline{A\overline{C}D})^* - (E\overline{A\overline{C}D})^*$$

$$= A_3(1) - 2A_3 \left( \frac{c - bc}{bc - b + 1} \right)$$

$$- A_3 \left( \frac{bc - b + 1}{(1 - c)(1 - b)} \right) + A_3 \left( \frac{c}{1 + abc} \right)$$

$$- A_3 \left( \frac{(1 - c)(1 - b) - h(ac - c + 1)}{-b(bc - b + 1)} \right) + A_3 \left( \frac{(1 - c)(1 - b)^2}{-b(bc - b + 1)} \right)$$

$$+ A_3 \left( \frac{abc}{1 + abc} \right) - A_3 \left( \frac{(1 - c)(1 + abc)}{-abc^2} \right)$$

$$+ A_3 \left( \frac{-bc^2}{(1 - c)(bc - b + 1)} \right)$$

$$- A_3 \left( \frac{bc}{bc - b + 1} \right) - A_3 \left( \frac{ac - c + 1}{ac} \right) + A_3 \left( \frac{ac - c + 1}{ac(1 - b)} \right)$$

$$- A_3 \left( \frac{c - 1}{ac} \right) + A_3 \left( \frac{c - 1}{abc} \right) - A_3 \left( \frac{bc}{(1 - c)(1 - b)} \right).$$

(2) $[L; \breve{g} M]$. By abuse of notation

$$\breve{g} M = (ABCD) = \begin{pmatrix} 1 + abc & bc - b + 1 & 1 & 1 + abc \\ 1 + abc & bc - b + 1 & 1 & ac - c + 1 \\ 1 & 1 - b & 1 & 1 \\ 1 & 1 - b & 0 & 1 \end{pmatrix}.$$

Let $I = (1 - b, 0, 1, 0) = \triangle(BCD) \cap l_0 l_2$. Cutting $L$ into $L' = (l_0 l_1 l_3)$ and $L'' = (l_1 l_2 l_3)$ and using additivity we get

$$[L; \breve{g} M] = [L'; \breve{g} M] + [L''; \breve{g} M].$$
Put
\[ \tilde{g}' = \begin{pmatrix} 1/b & 0 & (b-1)/b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \]
and
\[ g'' = \begin{pmatrix} 1/b & 0 & (b-1)/b \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \text{PGL}_3(F). \]

Then
\[ [L; \tilde{g}M] = [g''E; \tilde{g}^*gM] + [g''E; g''gM] = [L; g''gM] - [L; g''gM]. \]

(i) \( g''gM \). By abuse of notation
\[ g''gM = (ABCD) = \begin{pmatrix} ac + 1 & 1 - b + c & 1 & ac + 1 \\ 1 + abc & bc - b + 1 & 1 & ac - c + 1 \\ 1 & 1 - b & 1 & 1 \\ 1 & 1 - b & 0 & 1 \end{pmatrix}. \]

Let \( E = (ac + 1, ac - c + 1, ac + 1 - (c/(1-b)), 1) = l_2D \cap BC \). Then \( [L; g''gM] = [L; (EACD)] - [L; (EACD)] \) by looking at the orientations (see the left picture of Fig. 20).

To treat \([L; (EACD)]\) let \( F = Cl_3 \cap EA, G = l_1F \cap AD \) and \( H = l_1F \cap ED \) (see the middle picture of Fig. 20). Because \((EACD), (CFHE), and\)

![FIG. 20. Second method: part (2)(i) (also for (2)(ii)).](image-url)
(CFHD) are oriented positively while (CFGA) and (CFGD) are oriented negatively we have

$$\mathcal{EACD}' = (CFHE) - (CFGA) - (CFHD) + (CFGD)$$

$$= -A_{1,2} \left( \frac{1}{1-b} \cdot \frac{c}{ac+1} \right) - A_{1,2} \left( 1 - \frac{c}{ac+1} \right)$$

$$+ A_{1,2} \left( a, \frac{c}{ac+1} \right) + A_{1,2} \left( 1 - \frac{a}{ac+1} \right).$$

To treat \([L; (EABD)]\) we let

$$P = Bl_3 \cap EA, \, Q = l_1 P \cap ED \text{ and } R = l_2 P \cap AD \text{ (see the right picture of Fig. 20). Because (EABD), (BPRD), and (BPQE) are oriented positively while (BPRA) and (BPQE) are oriented negatively we have}$$

$$\mathcal{EABD}' = (BPRD) - (BPRA) - (BPQD) + (BPQE)$$

$$= A_{1,1,1} \left( \frac{-c(1-b)}{bc-b+1}, \frac{-c(ab-a+1)}{(ac+1)(1-b)} \right)$$

$$- A_{1,1,1} \left( \frac{(1-b)^2}{bc-b+1}, \frac{-c(ab-a+1)}{(ac+1)(1-b)} \right)$$

$$+ A_{1,1,1} \left( \frac{bc-b+1}{bc-b+1}, \frac{-c(1-b) - c(ab-a+1)}{(ac+1)(1-b)} \right)$$

$$- A_{1,1,1} \left( \frac{bc-b+1}{(1-b)^2}, \frac{-c(1-b) - c(ab-a+1)}{(ac+1)(1-b)} \right).$$

Hence by Lemma 3.21 and Lemma 3.24

$$- [L; g'gM] = (\mathcal{EACD}') - (\mathcal{EABD}')$$

$$= 2A_3(1) - A_3(a) + A_3(1-b) - A_3 \left( \frac{1+abc}{1-b} \right) - A_3 \left( \frac{abc}{1+abc} \right)$$

$$+ A_3 \left( \frac{-ac(1-b)}{1+abc} \right) + A_3(a(ac-c+1)) - A_3(ac-c+1)$$

$$- A_3(-ac) - A_3 \left( \frac{-c}{ac-c+1} \right) + A_3 \left( \frac{-c}{1-b} \right) - A_3 \left( \frac{-bc}{1-b} \right)$$

by using $$A_3(t) + A_3(1-t) + A_3(1-t^{-1}) = A_3(1)$$ for $$t = (1+abc)/(1-b).$$
By abuse of notation
\[ g^* M = (ABCD) = \begin{pmatrix} ac + 1 & 1 - b + c & 1 & ac + 1 \\ 1 + abc & bc - b + 1 & 1 & ac - c + 1 \\ 1 + abc & bc - b + 1 & 1 & 1 + abc \\ 1 & 1 - b & 0 & 1 \end{pmatrix} \]

Let \( E = (ac + 1, ac - c + 1, ac - c + 1, 1) = l_2 D \cap BC \). Then \([ L; g^* g M ] = [ L; (EABD) ] - [ L; (EACD) ]\) (see the left picture of Fig. 20).

To treat \([ L; (EACD) ]\) let \( F = Cl_3 \cap EA, G = l_2 F \cap AD \) and \( H = l_1 F \cap ED \) (see the middle picture of Fig. 20). Then
\[
(EACD)^* = (CFHE) - (CFGA) - (CFHD) + (CGFD)
\]
\[ = -A_{1,2} \left( \frac{c}{ac + 1} \right) - A_{1,2} \left( \frac{ac(1 - b)}{ac + 1} \right) + A_{1,2} \left( \frac{a(1 - b)}{ac + 1} \right) - A_{1,2} \left( \frac{(ab - a + 1)}{(ac + 1)(1 - b)} \right) \]

To treat \([ L; (EABD) ]\) we let \( P = Bl_3 \cap EA, Q = l_1 P \cap ED \) and \( R = l_2 P \cap AD \) (see the right picture of Fig. 20). Then
\[
(EABD)^* = -(BPRA) + (BPDR) - (BPQD) + (BPQE)
\]
\[ = -A_{1,1,1} \left( \frac{(1 - b)^2}{bc - b + 1} \right) + A_{1,1,1} \left( \frac{-c}{bc - b + 1} \right) + A_{1,1,1} \left( \frac{1 - bc}{bc - b + 1} \right) - A_{1,1,1} \left( \frac{-c}{bc - b + 1} \right) \]

Hence by Lemma 3.21 and Lemma 3.24
\[
[L; g^* g M ] = (EABD)^* - (EACD)^*
\]
\[ = -A_3 (ab - a + 1) - A_3 \left( \frac{1 + abc}{(a(1 - b)(ac - c + 1))} \right) + A_3 \left( \frac{ac - c + 1}{1 + abc} \right) - A_3 \left( \frac{(ab - a + 1)}{ab - a} \right) \]
by using $A_4(t) + A_3(1-t) + A_3(1-t^{-1}) = A_3(1)$ for

$$t = \frac{(1-b)^2}{bc-b+1} \cdot \frac{c(1-b)}{ac+1} \cdot \frac{1+abc}{b(ac+1)} \cdot \frac{1+abc}{ac-c+1} \cdot \frac{ac-c+1}{1+abc} \cdot \frac{e(1+abc)}{(ac-c+1)(1-b)}.$$

Finally we can put all parts together to get

$$[L; M] = [L; \hat{g} \hat{g} M] - [L; \hat{g}^* \hat{g} M] - [L; \hat{g}^* \hat{g} M] + [L; \hat{g}^* \hat{g} M]$$

$$= 2A_3 \left( \frac{1+abc}{ac-c+1} \right) + A_3(a-ab) - A_3 \left( \frac{a(1+abc)(1-b)}{ac-c+1} \right)$$

$$+ A_3(1-c) - A_3 \left( \frac{(1-c)(1+abc)}{ac-c+1} \right)$$

$$+ A_3 \left( \frac{-bc(ac-c+1)}{(1+abc)(1-c)(1-b)} \right) - 2A_3 \left( \frac{(1-c)(1-b)}{bc} \right)$$

$$- 2A_3 \left( \frac{c-bc}{bc-b+1} \right) - A_3 \left( \frac{bc-b+1}{(1-c)(1-b)} \right)$$

$$+ A_3 \left( \frac{c}{1+abc} \right) - A_3 \left( \frac{(1-c)(1-b)}{-(bc-c+1)} \right)$$

$$+ A_3 \left( \frac{(1-c)(1-b)(bc-b+1)}{-b(bc-c+1)} \right) - A_3 \left( \frac{(1-c)(1+abc)}{-abc} \right)$$

$$+ A_3 \left( \frac{-bc^2}{(1-c)(bc-b+1)} \right) - A_3 \left( \frac{bc}{bc-b+1} \right) - A_3 \left( \frac{ac-c+1}{ac} \right)$$

$$+ A_3 \left( \frac{ac-c+1}{ac(1-b)} \right) - A_3 \left( \frac{c-1}{ac} \right) + A_3 \left( \frac{c-1}{abc} \right) - A_3(a) + A_3(1-b)$$
\[-A_3\left(\frac{1 + abc}{1 - b}\right) + A_3\left(\frac{-ac(1 - b)}{1 + abc}\right) + A_3(a(ac - c + 1))\]
\[-A_3(ac - c + 1) - A_3(-ac) - A_3\left(\frac{-c}{ac - c + 1}\right) - A_3\left(\frac{-bc}{1 - b}\right)\]
\[-A_3(ab - a + 1) - A_3\left(\frac{1 + abc}{ab(1 - b)(ac - c + 1)}\right) - A_3\left(\frac{ab - a + 1}{ab - a}\right)\]
\[+ A_3\left(\frac{-b(ac - c + 1)}{bc - b + 1}\right) + A_3\left(\frac{-c(1 + abc)}{(ac - c + 1)(1 - b)}\right)\]
\[+ A_3\left(\frac{1 + abc}{bc - b + 1}\right) + A_3(b).\]

Note that from \(A_3(x) = A_3(x^{-1})\) we can define \(A_3(\infty) = 0\). Now take \(b = 1\) and we get from the first method
\[[L; M] = -\lambda_3(\{a\} + \{-ac\} + \{c\} + \{ca - c + 1\} + \left\{\frac{ca - c + 1}{ca}\right\} - \left\{\frac{ca - c + 1}{a}\right\} - \{1\}).\]

and from the second method
\[[L; M] = \lambda_3(\{a\} + \{-ac\} + \{c\} + \{ca - c + 1\} + \left\{\frac{ca - c + 1}{ca}\right\} - \left\{\frac{ca - c + 1}{a}\right\} - \{1\}).\]

Thus
\[\lambda_3(R_3(1, a, c)) = 0\]

which implies Eq. (13).

Remark 3.26. In fact, when \(b = 1\) the second method fails because non-admissible pairs appear. However, one can modify the ideas used in the second method to get the above result. The calculation is much simpler than the general case where \(b \neq 1\). We omit the details.

For \(b \neq 1\), subtracting the expression of \([L; M]\) we get by the second method from the expression by the first method we have
\[\lambda_3\left(R_3\left(\frac{ac - c + 1}{1 + abc}, \frac{b}{bc - b + 1}, \frac{c}{c - 1}\right)\right) + \lambda_3\left(R_3\left(\frac{1}{c}, 1 - b, -ca\right)\right) - \lambda_3\left(R_3\left(1, \frac{-bc}{1 - b(1 - c)}, \frac{(1 - b)(1 - c)}{bc - b + 1}\right)\right) = 0.\]
Using substitution $x = 1/c$, $y = 1 - b$ and $z = -ca$ we have modulo prisms

$$\lambda_3(R_3(x, y, z)) + \lambda_3\left(R_3\left(\frac{zx - x + 1}{y - x(1 - z + yz)}, \frac{y - 1}{1 - x}\right)\right) = 0.$$  

Using the fact that $R_3(x, y, z) = R_3(y, z, x) = R_3(z, x, y)$ and the relation $A_3(t) + A_3(1 - t) + A_3(1 - t^{-1}) = A_3(1)$ we finally get

$$\lambda_3(R_3(x, y, z)) = 0.$$  

This concludes the proof of Proposition 3.15.

3.6. Injectivity of $l^n_B$: $B_n \rightarrow A_n/P_n$, $n = 2, 3$

The injectivity follows immediately from the following

**Proposition 3.27.** $a_n \cdot l_n = \text{id}$ for $n = 2, 3$.

**Proof.** By definition, $l_2([x]_2)$ is given by (modulo prisms) $A_2(x) = [L; M]$ where we take $L$ to be the standard simplex and $M_0 = \{t_1 + t_2 = t_0\}$, $M_1 = \{t_1 = t_0\}$ and $M_2 = \{t_2 = xt_0\}$. Then

$$a_2(l_2([x]_2)) = a_2([L; M]) = [r(L_1 | L_2; M_0, M_2)],$$

because all other terms are zero.

By definition, $l_3([x]_3)$ is given by (modulo prisms) $-A_3(x) = [L; M]$ where $L$ is the standard simplex and $M_0 = \{t_2 = t_5\}$, $M_1 = \{t_3 = xt_0\}$, $M_2 = \{t_1 = t_0\}$, and $M_3 = \{t_1 + t_2 = t_0\}$. Then

$$a_3(l_3([x]_3)) = a_3([L; M]) = \frac{1}{4}a'_3([L; M]) - \frac{1}{4}a''_3([L; M]).$$

It is easy to see that all terms in $a'_3([L; M])$ is zero except

$$r_3(L_1 | L_0, L_2, L_3; M_0, M_1, M_3)$$

$$= r_3\begin{bmatrix} 1 & 0 & 0 & 0 & x & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix} = r_3\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -x \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} = -6[x]_3$$

by skewsymmetry of $r_3$ and Corollary 3.7. Hence

$$a'_3([L; M]) = -r_3(L_1 | L_0, L_2, L_3; M_0, M_1, M_3) = 6[x]_3.$$  

Now let us look at $a''([L; M])$. All the $\mu_3$-terms are zero except

$$\mu_3(L_0, M_1) = \mu_3(L_0, M_2) = \mu_3(L_1, M_2) = \mu_3(L_3, M_1) = -\{1\}_3$$
by Lemma 3.6. Taking the appropriate signs into consideration we see that these four terms cancel out each other in $a^d([L; M])$. Therefore

$$a_3 \otimes l_3(x^3) = \frac{1}{2} a_3^d([L; M]) = x^3.$$

This finishes the proof of our proposition.

### 3.7. Surjectivity of $l_n$: $B_n \to A_n/\Pi_n$, $n = 2, 3$

In this section we shall prove

**Theorem 3.28.** For $n = 2, 3$ the maps $l_n: B_n \to A_n/\Pi_n$ are surjective.

**Proof.** We only need to show $\lambda_2$ and $\lambda_3$ are surjective. The surjectivity of $\lambda_2$ is proved in [2]. We sketch here another geometric proof. Take any $[L; M] \in A_2$. By projective invariance we may assume $L$ is the standard coordinate simplex. By skewsymmetry and additivity we can assume that at most one vertex of $M$ lies on some side of $L$, say $L_1$. Then by additivity (i.e., by cutting $M$) if necessary we can further assume either $M$ has two sides parallel to the coordinate axes or a vertex of $M$ lies on $L_1$ and its opposite side is parallel to $L_2$. In the first case we can see easily that $[L; M]$ is given by a difference of two dilogarithms modulo a rectangle. In the second case, by drawing the smallest rectangle to bound $M$ we see that $M$ is negative to the sum of two dilogarithms modulo the rectangle.

The surjectivity of $\lambda_3$ follows from Lemmas 3.30, 3.24, 3.21, and 3.19.

The following generalization of [3, Corollary 3.9.2] is obvious.

**Corollary 3.29.** The Aomoto trilogarithms may be represented by classical trilogarithms, products of classical dilogarithm with logarithm, and products of logarithms.

A more precise statement of this corollary has already been found by Goncharov [12, Theorem 3.9].

**Lemma 3.30.** Modulo prisms $A_3 = A(3)$.

**Proof.** Let $[L; M] \in A_3$ be an arbitrary admissible pair. It suffices to show that it is a sum or difference of pairs in the form of $A_{1,1,1}$, $A_{1,2}$, $A_{2,1}$ and $A_3$ modulo prisms.

Using additivity property on both $L$ and $M$ we can reduce the problem to finitely many elementary cases. Then we prove the lemma for each of the cases.

First we need some additional terminology. For two distinct points $p$ and $q$ we denote by $pq$ the line through them. For coplanar points $p_1, \ldots, p_r$ we denote by $\triangle(p_1 \cdots p_r)$ the plane through them. Let $M = (M_0, \ldots, M_n)$ be
a nondegenerate simplex in $\mathbb{P}_F^n$. For a subset $\{j_1, \ldots, j_t\}$ of $\{0, \ldots, n\}$ we write $M_{j_0 \ldots j_t} = \cap_{j=1}^t M_{j_t}$. We call a chain of inclusions of the faces $M_{i_0} \subset M_{i_1} \subset \cdots \subset M_{i_t} \subset M_n$ a flag of $M$ (there are $(n+1)!$ different flags in $M$). We denote $m_i$, the vertex of $M$ facing $M_{i}$ (a lower Roman letter faces a capital one).

Let $[L; M] \in A_3$ be a nondegenerate admissible pair. By additivity on both $L$ and $M$ we may assume that if $(L; M)$ is not a generic pair then the nongeneric conditions occur in only one flag of $L$ and only one flag of $M$. Hence we have the following nine cases to consider,

(I) $(L; M)$ is in generic position;  
(II) $m_0 \in L_2$;  
(III) $m_0 \in l_0 l_1$;  
(IV) $m_0 \in l_0 l_1$ and $m_0 m_3 \subset L_2$;  
(V) $m_0 \in l_0 l_1 \subset M_3$;  
(VI) $l_2 \in M_4$;  
(VII) $l_2 \in m_2 m_3$;  
(VIII) $l_1 \in m_1 m_3$ and $l_0 l_1 \subset M_5$;  
(IX) $l_2 \in m_2 m_3 \subset L_1$,

where the conditions in each of the above cases are the only nongeneric conditions for the pair in that case. There are three more cases which are not in the above list and which involve only one flag of $M$ and only one flag of $L$:

(X) $m_0 m_1 \cap l_0 l_1 \neq \emptyset$;  
(XI) $m_0 m_1 \in L_2$;  
(XII) $l_0 l_2 \in M_1$.

Case (X) can be reduced to case (III): Let $E = m_0 m_1 \cap l_0 l_1 \neq \emptyset$. But by additivity $M$ is a sum (or difference) of $M'$ and $M''$ where we get $M'$ (resp. $M''$) by replacing $m_0$ (resp. $m_1$) by $E$. Then $(L; M)$ is a sum (or difference) of two pairs in the case (III). Similarly, (XI) can be reduced to case (IV) while (XII) to (IX).

Now we proceed to prove the lemma for the cases (I)–(IX). First we have the following criterion of $(L; M) \in A_3$:

There are three non-coplanar edges of $M$ which have no common point and each of them includes exactly one vertex of $L$.

We adopt the following conventions in all of the pictures:

— edge;
• invisible edge;
— cutting line, also the edge of $L$ if it’s the standard simplex;
— extension of edge;
—• connection between $M$-point to $L$-point;
— other auxiliary line, intersections of planes, etc.
(1). \((L; M)\) Is a Generic Pair. Step (1). Let \(A = l_2m_2 \cap M_2\) and cut \(M\) into three parts by the planes \(m_i, A\) for \(i = 0, 1, 3\) (see the left picture in Fig. 21). We may assume that the only nongeneric condition is given by a inclusion of a vertex of \(L\) in an edge of \(M\), say, 
\[l_2 \in m_2m_3.\]

Step (2). Cut \(M\) by the plane \(P = \triangle(l_1l_2m_3m_4)\). Write \(B = m_0m_1 \cap P\) and \(C = m_2m_3 \cap l_1B\) (see the right picture in Fig. 21). Thus \(M\) is cut into four parts and we may assume that the only nongeneric conditions are given by the inclusions of two vertices of \(L\) in two adjacent edges of \(M\), respectively, say
\[l_1 \in m_1m_2, \quad l_2 \in m_2m_3.\]

Step (3). Let \(X = L_1 \cap m_0m_3\) and \(Y = L_1 \cap m_1m_3\) and \(Z = XY \cap l_0l_3\). Cutting \(L\) into two parts by \(\triangle(l_1l_2Z)\): \(L' = (l_0Zl_1)\) and \(L'' = (Zl_1l_2l_3)\) (see the left picture in Fig. 22). Notice that in both \(L'\) and \(L''\) the only nongeneric conditions are
\[l_1 \in m_1m_2, \quad l_2 \in m_2m_3, \quad Z \in M_2.\]

Step (4). We only need to deal with \(L'\) because \(L''\) has exactly the same conditions. Let \(E = \triangle(m_1m_2l_1Z) \cap l_2m_0\) and \(F = \triangle(m_3l_1Z) \cap l_2m_0.\)

FIG. 21. Generic case steps (1) and (2).

FIG. 22. Generic case steps (3) and (4).

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Let $G$, $H$ and $I$ be the intersection of $M_0$ with $FZ$, $Zm_0$ and $EZ$ respectively (see the right picture in Fig. 22 where $Z$, $l_1$ and $l_2$ are at infinity so that two lines intersecting at any one of them appear to be parallel). Then we get

$$(L'; M) = (L'; m_0Hm_1) + \text{Prism}(FGm_3Elm_2)$$

$- (L'; m_0Elm_2) - (L'; m_0Hgm_3) - (L'; m_0FGm_3) \in H + A_{1,1,1},$

(28)

using criterion (*). There is an easier solution to be given in (IX). But it does not work for cases (II) and (III).

II. $m_0 \notin L_1$. Each of the four steps in (I) works well. We observe that in each step we may produce some pairs appearing in the corresponding step of case (I). This observation applies to the treatment of all the other cases as well. For example, in case (III) we may need a result in cases (I) and (II).

Looking at Fig. 22 we see that the only difference in this case lies in step (4): for the four simplices in Eq. (28) we have $(L'; m_0Elm_2)$, $(L'; m_0FGm_3) \in A_{2,1}$ and $(L'; m_0Hgm_3), (L'; m_0FGm_3) \in A_{1,2}$ because $m_0 \notin L_1$.

III. $m_0 \in l_0l_1$. We can use the same steps as in (I). The only difference appears in the final equation of step (4): all of the four pairs of simplices belong to $A_3$ because $m_0 \in l_0l_1$.

IV. $m_0 \in l_0l_1$ and $m_0m_3 \subset L_2$. Steps (1) and (2). By carrying out steps (1) to (2) of (I) we now may assume

$$l_1 \in m_1m_2, \quad l_2 \in m_2m_3, \quad m_0 \in l_0l_1, \quad m_0m_3 \subset L_2.$$ 

Step (3). Let $Z = l_0l_1 \cap m_0m_3$ and cut $L$ into two parts: $L' = (l_0Zl_1l_2)$ and $L'' = (Zl_1l_2l_3)$ (see Fig. 23). Then we see that both $(L'; M)$ and $(L''; M')$ satisfy the criterion (*) and in fact they belong to $A_3$ and $A_{2,1}$, respectively. Essentially, this step is a variant of step (3) because $Z = l_0l_1 \cap M_2$. We don’t need step (4) since $Z \notin m_0m_3$ already.
Step (1). Carrying out steps (1) of (I) we let $A = l_2m_2 \cap M_2$. Cut $M$ into three parts. $(Am_0m_2m_3)$ satisfies the conditions of case (III) after step (1) while $(Am_1m_2m_3)$ satisfies the conditions of case (I) after step (1) (see Fig. 24). Hence we only need to consider the simplex $(Am_0m_1m_2)$.

Step (2). $(Am_0m_1m_2)$ can be cut into two more parts by $\triangle(l_1m_2AB)$ where $B = \triangle(l_1m_2A) \cap m_0m_1$. So $B = l_1m_2 \cap m_0m_1$ because $l_1 \in M_3$. Then $(ABm_1m_2)$ satisfies the condition

$$l_1 \in m_2B, \quad l_2 \in m_2A, \quad l_0l_1 \subset \Delta m_1m_2B,$$

while $(ABm_0m_2)$ satisfies

$$l_1 \in m_2B, \quad l_2 \in m_2A, \quad m_0 \in l_0l_1 (\subset \Delta m_0m_2B).$$

We put $l_0l_1 \subset \Delta m_0m_2B$ in parentheses because it can be derived from $l_1 \in m_2B$ and $m_0 \in l_0l_1$.

Step (3). First note that the same configuration as $(ABm_0m_2)$ appears in the case (III) after step (2) ($A$ corresponds to $m_3$ while $B$ to $m_1$ there; see the left picture in Fig. 24). Therefore we only need to look at $(ABm_1m_2)$ (see the right picture in Fig. 24). Let $C = \triangle(l_0m_1A) \cap m_2B$ and
cut \((ABm_1m_3)\) into two parts by \(\triangle(l_0m_1AC)\). Then \((ACm_1m_3)\) satisfies the
criterion (*) and \((ACm_1B)\) satisfies the two conditions in case (I) after step
(2).

(VI). \(l_2 \in M_1\). Let \(E = l_2m_1 \cap m_0m_2\) and cut \(M\) into \((m_1m_2m_3E)\) and
\((m_0m_1m_3E)\). So we may assume
\[ l_2 \in m_1m_3. \]
This is the condition after step (1) of (I). See Fig. 25.

(VII). \(l_2 \in m_2m_3\). This is the condition after step (1) of (I).

(VIII). \(l_2 \in m_1m_2\) and \(l_0l_1 \in M_1\). Step (1). Let \(E = m_0m_2 \cap l_0l_1\). We
see that \(M\) can be cut into two parts: \((m_0m_1m_3E)\) and \((m_1m_2m_3E)\). Now
\((m_0m_1m_3E)\) belongs to case (V) and \((m_1m_2m_3E)\) satisfies
\[ E \in l_0l_1 \subset M_3, \quad l_1 \in m_1m_2. \]
Step (2). Let \(A = l_2m_3 \cap M_0\) and cut \((m_1m_2m_3E)\) into three parts
(see Fig. 26). Then \((m_1Am_1E)\) appears after step (1) in (III), \((m_2Am_1E)\)
after step (2) in (III), while \((m_2Am_1m_3)\) after step (2) in (I).

(IX). \(l_2 \in m_2m_3 \subset L_1\). Because the condition includes the one in case
(1) after step (1) (namely, \(l_2 \in m_2m_3\)) we may follow steps (2) to (4) in (I).
without any problem. The left picture in Fig. 27 shows its initial condition. After step (3) we may assume

\[ l_1 \in m_2 m_2, \quad l_2 \in m_2 m_3 \leq L_1, \quad l_3 \in M_2. \]

Now we can use a variant of step (4) in case (I) to finish the reduction procedure. See the right picture in Fig. 27 where \( m_2 m_3 \leq L_1 \). Note that 
\( (m_0 E G m_3), (m_0 E G m_2) \in A_{2,1}, (m_0 E F m_2) \in A_{1,2} \) and \( (m_0 E F m_1) \in A_{1,1,1} \).

This concludes the proof of Lemma 3.30. 

4. A BYPRODUCT: AN ISOMORPHISM OF TWO COMPLEXES

The results in Section 3 allow us to prove

**Theorem 4.1.** Let \( F \) be a field, \( A_n = A_n(F) \) and \( B_n = B_n(F) \). Then the following diagram is commutative,

\[
\begin{array}{ccc}
A_3 & \xrightarrow{a_3} & (A_2 \otimes A_1) \oplus (A_1 \otimes A_2) \\
& \Downarrow (a_2 \otimes \tau) \circ (1 - \tau) & \downarrow \alpha \\
B_3 & \xrightarrow{\alpha_3} & B_2 \otimes F^x \\
& \Downarrow \alpha_3 \otimes \text{id} & \downarrow \bigwedge^3 r
\end{array}
\]

where \( r \) is the cross ratio and \( \tau(x \otimes y) = y \otimes x \).

**Proof.** By the result in [22] the coproducts \( v_{2,1} : A_3 \to A_2 \otimes A_1 \) and \( v_{1,2} : A_3 \to A_1 \otimes A_2 \) are defined. It is also easy to see that the top line in the diagram is a complex by the coassociativity of the coproducts.

By Theorem 3.28, \( A_3 \) is generated by prisms and \( A_3(r) \). We now only need to verify that the diagram commutes for these two kinds of objects in \( A_3 \). Now Proposition 3.27 shows that \( a_n(A_n(x)) = (-1)^n \{ x \} \) for \( n = 2, 3 \) and Proposition 3.8 says \( a_n \) sends prisms to zero for \( n = 2, 3 \). Therefore the theorem follows from (see [22])

\[
(v_{2,1} + v_{1,2})(A_3(x)) = -\left[ \frac{x^2}{2} \otimes (1 - x) \right] - [x \otimes A_2(x)].
\]

**Remark 4.2.** The corresponding result when \( n = 2 \) is proved in [2]. Goncharov [12] proves a weaker version of the above by replacing \( A_3 \) by the abelian group \( A_3^n \) generated by generic pairs in \( \mathbb{P}_F^3 \).
It is now easy to see that

**Corollary 4.3.** The two complexes

\[
A_3 / \Pi_3 \xrightarrow{\nu} (A_2 / \Pi_2) \otimes A_1 \xrightarrow{\omega} \bigwedge^3 A_1
\]

and

\[
B_3 \rightarrow B_2 \otimes \mathbb{F}^\times \rightarrow \bigwedge^3 \mathbb{F}^\times
\]

are isomorphic, where \(\nu(a) = v_{2.1}(a) - \tau \cdot v_{1.2}(a)\) and \(\nu'(x \otimes y) = v'_{1.1}(x) \wedge y\). Here \(v'_{1.1}(x)\) denotes the image of \(v_{1.1}(x)\) in \(A_1 \wedge A_1\).

**Proof.** It is straightforward to check that the first line is indeed a complex and is well defined. The rest follows from Theorem 4.1, Main Theorem 3.1, and [3, Main Theorem 2].

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