MOTIVIC COHOMOLOGY OF PAIRS OF SIMPLICES

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1. Introduction

Let F be a field and $n \geq 2$ be a positive integer. A simplex in the projective space $\mathbb{P}_F^n$ is an ordered set of hyperplanes $L = (L_0, \dots, L_n)$. A face of L is any non-empty intersection of the hyperplanes. A pair of simplices is admissible if they do not have common faces of the same dimension. It is a generic pair if all the faces of the two simplices are in general position.

In their seminal paper [2, 4] Beilinson *et al.* initiated the study of the motivic cohomology of the admissible pairs of simplices $(L; M)$ on $\mathbb{P}_F^2$ which is defined as a graded comodule $\bigoplus_{j=0}^2 H_{\text{mot}}^*(L; M)_j$ over a suitable graded algebra $A(F)_{\bullet, \mathbb{Q}} := A(F)_{\bullet} \otimes \mathbb{Q}$. The cohomology is the arithmetico-algebraic analog of the relative Betti cohomology and is defined via mysterious complexes formed by the geometric combinatorial data of the pair. The reason for this is that the graded pieces of the cohomology groups can actually be obtained from a spectral sequence converging to $H^*(\mathbb{P}_{\mathbb{C}}^n \setminus L, M \setminus L; \mathbb{Q})$ with degenerate E_2 -term. In this paper we propose a generalization of the above from $\mathbb{P}_F^2$ to $\mathbb{P}_F^n$.

Briefly speaking, the group $A_n(F)$ is generated by admissible pairs of simplices in $\mathbb{P}_F^n$, subject to a set of relations (see Definition 2.1 for the detail). The defining (double scissors congruence) relations reflect (conjecturally all of) the functional equations of Aomoto polylogarithms first studied in [1]. These groups are expected to form a Hopf algebra and are closely related to algebraic K -theory by the following conjecture [4, p. 550] (after slight reformulation).

CONJECTURE 1.1. The restricted coproduct induces a complex

$$A_{>0} \longrightarrow A_{>0} \otimes A_{>0} \longrightarrow A_{>0} \otimes A_{>0} \otimes A_{>0} \longrightarrow \dots$$

whose graded n -piece

$$A_n \longrightarrow \bigoplus_{k=1}^{n-1} A_k \otimes A_{n-k} \longrightarrow \dots$$

provides the isomorphism

$$H_{(n)}^i(A_{\bullet}, \mathbb{Q}) \cong \text{gr}_n^{\gamma} K_{2n-i}(F)_{\mathbb{Q}},$$

where γ stands for the γ -filtration of K -groups.

Beilinson *et al.* proved this conjecture up to $K_3(F)$ in [2, 4]. We now have a lot of evidence up to $\text{gr}_4^{\gamma} K_7(F)$ (see [10, 14]).

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According to the Tannakian formalism the category $\text{MTM}(F)$ of mixed Tate motives over a field F is supposed to be equivalent to the category of graded modules over a certain graded commutative Hopf algebra $\mathcal{A}_\bullet$ (see [3] and [9, Chapter 3]). Therefore the Ext groups in the category $\text{MTM}(F)$ are isomorphic to the cohomology of the Hopf algebra $\mathcal{A}_\bullet$. Beilinson *et al.* conjecture that $A_\bullet := \bigoplus_{n \geq 0} A_n$ is isomorphic to $\mathcal{A}_\bullet$ and therefore the graded object $A_\bullet \otimes \mathbb{Q}$ should have a Hopf algebra structure over $\mathbb{Q}$. This is the primary motivation to study the groups A_n in this paper.

In [13] by using a geometric combinatorial method we analyzed the generic part $A_\bullet^0$ of $A_\bullet$ (see page 316 for the precise definition of $A_\bullet^0$). In particular, we proved that the coproduct is well defined on the generic part. However, the proofs are complicated due to their computational nature. In this paper we will explore a different path by applying the theory developed in [2, 3] to study the motivic cohomology of all the admissible pairs of simplices, not only the generic ones. It follows from the general theory of framed mixed Hodge structures that the coproduct map on A_n should exist. This idea will then be used to find the *explicit* definition of the coproduct map on A_n for $n \leq 4$.

We now describe the content of this paper in some detail. Section 2 is a brief review of the double scissors congruence groups and the framed Hodge–Tate structures. In §3, for a non-degenerate admissible pair $(L; M)$ in $\mathbb{P}_{\mathbb{C}}^n$ we explicitly define a linearly constructible motivic perverse sheaf $\mathcal{S}(L; M)$ without monodromy in the sense of [3] and prove that $\mathcal{S}(L; M)$ provides the Hodge–Tate structure of the relative Betti cohomology $H^*(\mathbb{P}_{\mathbb{C}}^n \setminus L, M \setminus L; \mathbb{Q})$ when L and M are in general position. Essentially $\mathcal{S}(L; M)$ is the E_0 -term of a spectral sequence which converges to $H^*(\mathbb{P}_{\mathbb{C}}^n \setminus L, M \setminus L; \mathbb{Q})$. It is shown in [3] that $\mathcal{S}(L; M)$ is uniquely determined by a 2-dimensional diagram which satisfies some compatibility conditions. We denote this diagram also by $\mathcal{S}(L; M)$. Let $\mathbf{K}_{2j}^{\bullet, \bullet}$ be the bicomplex inside $\mathcal{S}(L; M)$ corresponding to the $2j$ th weight graded piece and let $\mathbf{K}_{2j}^\bullet$ be its total complex. Then we have the following result.

THEOREM 3.10. *Let $(L; M)$ be a pair of simplices in $\mathbb{P}_{\mathbb{C}}^n$ in general position. Then the relative Betti cohomology*

$$H^{n+*}(\mathbb{P}^n \setminus L, M \setminus L; \mathbb{Q}) \cong \bigoplus_{j=0}^n H^*(\mathbf{K}_{2j}^\bullet(L; M)).$$

The benefit of the diagram $\mathcal{S}(L; M)$ is that it can be described in purely geometric combinatorial terms. For a general field F we can therefore recover the motivic cohomology $H_{\text{mot}}^*(L; M)$ using a similarly constructed 2-dimensional diagram to define

$$H_{\text{mot}}^q(L; M) := \bigoplus_{j=0}^n H^q(\mathbf{K}_{2j}^\bullet(L; M))$$

graded by the weight. When $F = \mathbb{C}$ this is consistent with Theorem 3.10. The main result about the structure of the motivic cohomology $H_{\text{mot}}^*(L; M)$ is given by the following.

MAIN THEOREM 3.25. *The complex $\mathbf{K}_{2j}^\bullet$ is concentrated on degrees $[j - n, j]$ and is exact everywhere except at $\mathbf{K}_{2j}^0$. Moreover, $H^0(\mathbf{K}_0^\bullet(L; M)) \cong H^0(\mathbf{K}_{2n}^\bullet(L; M)) \cong \mathbb{Q}$ and $H^0(\mathbf{K}_{2j}^\bullet(L; M))$ has dimension at most $\binom{n}{j}^2$ for $0 < j < n$. The equality holds when $(L; M)$ is a generic pair and there exists a canonical choice of basis in this case.*

In §4, from the above explicit construction of the motivic cohomology we are able to recover the definition of the coproduct on the generic part $A_\bullet^0$ which is in agreement with [3]. One of the advantages of our approach in this generic case is that it follows directly from our definition that the coproduct is well defined although it is not obviously so at a first glance.

For A_2 our general theory also enables us to handle all the non-generic cases in §5 which greatly simplifies the computation for pairs of triangles in [2].

One of the most important applications of our motivic cohomology theory is given in §6. There, we first reduce the problem of defining the coproduct on A_3 to the generic case and finitely many non-generic cases. Then for each of them we are able to define the coproduct explicitly. Having the clear picture of A_3 in our mind we are able to generalize to A_3 a few deep results about A_2 in another paper [14].

Using an argument similar to that used in dealing with A_3 we can further define the coproduct explicitly on all of A_4 . Nevertheless, in the last section, we content ourselves with only two crucial examples of A_4 to show how our general theory can be applied in these cases. One example is a part of the classical tetralogarithmic pair of simplices. The other is essentially the only obstacle to the construction of the Aomoto tetralogarithm by using classical tetralogarithms and products of polylogarithms of lower weights.

This paper grew out of the attempt to give a conceptually clearer solution to the problem of defining the coproduct on A_n than the one presented in my thesis. My approach to the relative motivic cohomology in §3.2 is inspired by several conversations with C.-L. Chai to whom I want to express my hearty gratitude. The anonymous referee provided some very helpful comments which greatly improve the exposition of the paper.

2. Preliminaries and notation

We shall begin with some notation. Let F be a field. A *simplex* in the projective space $\mathbb{P}_F^n$ is an ordered set of hyperplanes $L = (L_0, \dots, L_n)$. It is *non-degenerate* if the intersection of all the hyperplanes L_i is empty. A *face* of L is a non-empty intersection of some ordered hyperplanes. For example, the face of $L_1 \cap L_2$ can be represented as $(L_1, L_2) = -(L_2, L_1)$. For any ordered index set

$$I = (i_1, \dots, i_k) \subset \llbracket n \rrbracket := \{0, 1, \dots, n\}$$

we set $L_I = \bigcap_{i \in I} L_i$ or $L_I = (L_{i_1}, \dots, L_{i_k})$ (ordered set of faces of L) depending on the context. We also adopt the convention that $L_{ij} = L_i \cap L_j$ and $L_{i,j} = (L_i, L_j)$ and so on. Given a non-degenerate simplex L we may choose the coordinate system $[t_0, \dots, t_n]$ in $\mathbb{P}_F^n$ such that $L_i = \{t_i = 0\}$ for $0 \leq i \leq n$. Such a simplex is called the *standard simplex*.

A pair of simplices is *admissible* if they do not have common faces of the same dimension. It is a *generic* pair if all the faces of the two simplices are in general position.

2.1. Double scissors congruence groups $A_n(F)$

These groups are first introduced in [3] and then modified in [2].

DEFINITION 2.1. Define $A_0(F) = \mathbb{Z}$. If $n > 0$ then $A_n(F)$ is the abelian group generated by admissible pairs of n -simplices $(L; M)$ subject to the following relations.

(R1) *Non-degeneracy.* We have $(L; M) = 0$ if and only if L or M is degenerate.

(R2) *Skew symmetry.* For every permutation σ of $\llbracket n \rrbracket$,

$$(\sigma L; M) = (L; \sigma M) = \text{sgn}(\sigma)(L; M),$$

where $\sigma L = (L_{\sigma(0)}, \dots, L_{\sigma(n)})$.

(R3) *Additivity in L and M .* For any $n+2$ hyperplanes $L_0, \dots, L_{n+1}$ and n -simplex M in $\mathbb{P}^n$,

$$\sum_{j=0}^{n+1} (-1)^j ((L_0, \dots, \widehat{L_j}, \dots, L_{n+1}); M) = 0$$

if every pair $((L_0, \dots, \widehat{L_j}, \dots, L_{n+1}); M)$ is admissible. A similar relation is satisfied for M .

(R4) *Projective invariance.* For every $g \in \text{PGL}_{n+1}(F)$,

$$(gL; gM) = (L; M).$$

Denote by $[L; M]$ the class of $(L; M)$ in $A_n(F)$.

CONVENTION. Sometimes we need to produce simplices by intersecting hyperplanes (for example, see the next proposition). Suppose $L_0, \dots, L_n$ and $M_0, \dots, M_n$ are hyperplanes in $\mathbb{P}^{n+1}$. If $N = L_i$ or $N = M_i$ for some i then we always throw away $(N|L; M) := ((L_0 \cap N, \dots, L_n \cap N); (M_0 \cap N, \dots, M_n \cap N))$ because they are not pairs of simplices.

The interested readers may find that we have replaced the relations in the *Trivial Intersection* axiom of [13, §2] by the above convention to make the definition logically better. They can also find a detailed analysis of this definition and the relations between our formulation and those of other authors. There, we proved the following.

PROPOSITION 2.2 (Intersection additivity). *For n hyperplanes $M_1, \dots, M_n$ and any $n+1$ hyperplanes $L_0, \dots, L_n$ in $\mathbb{P}^n$,*

$$\sum_{i=0}^n (-1)^i (L_i | (L_0, \dots, \widehat{L_i}, \dots, L_n); M) = 0$$

if every pair $(L_i | (L_0, \dots, \widehat{L_i}, \dots, L_n); M)$ is admissible on $L_i \cong \mathbb{P}^{n-1}$. A similar relation holds for M .

One often considers another family of groups $A_n^0(F)$ (called the *generic part* of $A_n(F)$) defined similarly to $A_n(F)$ except that

- (i) the generators $(L; M)$ are required to be generic pairs,
- (ii) in (R3), all $(L_0, \dots, \widehat{L_i}, \dots, L_{n+1}; M)$ are generic pairs.

2.2. Framed Hodge–Tate structures: a brief review

We will use [2] as our primary reference in this subsection.

Recall that a non-trivial Hodge–Tate structure is a mixed Hodge structure whose quotient of weight $-2k$ is isomorphic to some non-zero copies of $\mathbb{Z}(k)$. We say that H is an n -framed Hodge–Tate structure if it is equipped with a non-zero vector $v \in \mathrm{gr}_{2n}^W H$ and a non-zero covector $\tilde{v} \in (\mathrm{gr}_0^W H)^*$, that is, a pair of morphisms

$$v : \mathbb{Z}(-n) \longrightarrow \mathrm{gr}_{2n}^W H_1 \quad \text{and} \quad \tilde{v} : (\mathrm{gr}_0^W H)^* \longrightarrow \mathbb{Z}(0).$$

Consider the finest equivalence relation on the set of all n -framed Hodge–Tate structures for which $H_1 \sim H_2$ if there is a map $H_1 \rightarrow H_2$ respecting frames. For example, any n -framed Hodge–Tate structure is equivalent to some H with $W_{-2}H = 0$ and $W_{2n}H = H$. The equivalence classes form an abelian group $\mathcal{H}_n$ as follows: if two classes of $\mathcal{H}_n$ are represented by H and H' and their corresponding frames are denoted by $(v, \tilde{v})$ and $(v', \tilde{v}')$, then $[H] + [H']$ is represented by the Hodge–Tate structure $H \oplus H'$ with frame $(v + v', \tilde{v} + \tilde{v}')$ and $-[H]$ is the class of H with frame $(-v, \tilde{v})$ (or $(v, -\tilde{v})$). The tensor product of mixed Hodge structures induces the commutative multiplication $\mu : \mathcal{H}_i \otimes \mathcal{H}_j \rightarrow \mathcal{H}_{i+j}$.

One can define the coproduct

$$\nu = \bigoplus_{i+j=n} \nu_{i,j} : \mathcal{H}_n \longrightarrow \bigoplus_{i+j=n} \mathcal{H}_i \otimes \mathcal{H}_j$$

as follows. Let $(H, v, \tilde{v})$ be as above and $R \subset \mathrm{gr}_{2i}^W H$ be a lattice and $R^* \subset (\mathrm{gr}_{2i}^W H)^*$ the dual lattice. One can define the homomorphisms

$$\varphi : R \longrightarrow \mathcal{H}_i, \quad \psi : R^* \longrightarrow \mathcal{H}_j.$$

Namely, for $x \in R$, $\varphi(x)$ is the class of the sub-Hodge structure H , consisting of vectors in W_{2i} whose projection in $\mathrm{gr}_{2i}^W H$ is proportional to x , with framing $\{x, \tilde{v}\}$. For $y \in R^*$, $\psi(y)$ is the class of the quotient-structure $(H/W_{2i-1})/\ker y$ with framing $\{v, y\}$.

Let $\{e_j\}$ and $\{e^j\}$ be dual bases in R and R^* respectively. Then one defines the coproduct

$$\nu_{i,n-i}([H]) := \sum_j \varphi(e_j) \otimes \psi(e^j).$$

It is easy to see that μ and ν are compatible, that is,

$$\nu(\mu(a \otimes b)) = ((\mu \otimes \mu) \circ \tau)(\nu(a) \otimes \nu(b))$$

where $\tau(a \otimes b \otimes c \otimes d) = a \otimes c \otimes b \otimes d$. Set $\mathcal{H}_0 := \mathbb{Z}$. Then $\mathcal{H}_\bullet := \bigoplus_{n \geq 0} \mathcal{H}_n$ is a graded Hopf algebra with multiplication μ and coproduct ν .

The following result is the main reason for considering the Hopf algebra $\mathcal{H}_\bullet$. More details can be found in [3].

THEOREM 2.3. *The category of mixed $\mathbb{Q}$ -Hodge–Tate structures is canonically equivalent to the category of finite-dimensional graded $\mathcal{H}_{\bullet, \mathbb{Q}}$ -comodules.*

This theorem implies that the equivalence assigns to a Hodge structure H the graded comodules M^H and $M_n^H = \mathrm{gr}_{2n}^W(H)$ with $\mathcal{H}_\bullet$ -action $M^H \otimes (M^H)^* \rightarrow \mathcal{H}_\bullet$.

given by the formula

$$x_i \otimes y_j \longmapsto \{\text{class of mixed Hodge structures framed by } x_i, y_j\}.$$

Furthermore, the dual Hopf algebra $\mathcal{H}_{\bullet, \mathbb{Q}}^*$ is isomorphic to the universal enveloping algebra of the graded Lie algebra $\oplus L(F)$ where $L_n = (\mathcal{H}_n / \sum_{1 \leq k < n} \mathcal{H}_k \cdot \mathcal{H}_{n-k})_{\mathbb{Q}}^*$. Conjecturally one should be able to replace $\mathcal{H}_n$ by the double scissors congruence groups $A_n(F)$ everywhere in this subsection. The crucial piece of knowledge currently missing is how to define the coproduct *explicitly* in general.

3. Motivic cohomology of an admissible pair of simplices

This is the main theoretical part of this paper. We will apply its main results to a series of constructions in the next few sections.

3.1. The linearly constructible motivic perverse sheaf

Let F be an arbitrary field. Let $(L; M)$ be a non-degenerate admissible pair of simplices in $\mathbb{P}_F^n$. Denote the set of generic pairs of sub-simplices of $(L; M)$ by

$$\text{GP}(L; M) = \{(L_J; M_K) : I, J \subset [n], \text{codim}(L_J \cap M_K) = |J| + |K|\}.$$

For $0 \leq j, k \leq n$ write

$$\mathcal{L}_j \mathcal{M}_k = \{L_J \cap M_K : |J| = j, |K| = k, (L_J; M_K) \in \text{GP}(L; M)\}.$$

REMARKS 3.1. (i) We emphasize that one should understand $L_J \cap M_K$ as an ‘ordered’ intersection, for example, $L_{0,1} \cap M_1 = -L_{1,0} \cap M_1$.

(ii) By convention, $\mathcal{L}_j \mathcal{M}_0 = \mathcal{L}_j$, $\mathcal{L}_0 \mathcal{M}_k = \mathcal{M}_k$ and $\mathcal{L}_0 = \mathcal{M}_0 = \{\mathbb{P}^n\}$.

Following the idea of [3, §2.2] we first construct the *linearly constructible motivic perverse sheaves without monodromy* $\mathcal{S}(L; M)$ as follows. Briefly speaking, $\mathcal{S}(L; M)$ is the 2-dimensional diagram shown in Figure 1 consisting of

- (A) $\mathbb{Q}$ -vector spaces $\mathcal{S}_{i,d}$, with $i, d \in \mathbb{Z}$; and
- (B) linear maps (differentials) u and v .

FIGURE 1. Linearly constructible motivic perverse sheaf $\mathcal{S}(L; M)$.

(A) The space $\mathcal{S}_{i,d}$ is a direct sum $\bigoplus \mathcal{S}_i(N)$ where N ranges over d -dimensional projective subspaces of $\mathbb{P}_F^n$ which are in $\mathcal{L}_{s-d}\mathcal{M}_{n-s}$, with $d \leq s \leq n$. For all other projective subspaces N of $\mathbb{P}_F^n$ we set $\mathcal{S}_i(N) = 0$. Let $L_J \cap M_K \in \mathcal{L}_j\mathcal{M}_k$, with $j + k \leq n$. Then we set

$$\mathcal{S}_i(L_J \cap M_K) = \begin{cases} 0 & \text{if } i + k - j \neq n, \\ \langle (L_J \cap M_K) \rangle & \text{if } i + k - j = n, \end{cases}$$

where $\langle (L_J \cap M_K) \rangle$ is the $\mathbb{Q}$ -vector space generated by the symbol $(L_J \cap M_K)$. This definition forces us to identify

$$\text{sgn}((J, K) \rightarrow (J^{\text{inc}}, K^{\text{inc}}))(L_J \cap M_K) = \text{sgn}((\tilde{J}, \tilde{K}) \rightarrow (\tilde{J}^{\text{inc}}, \tilde{K}^{\text{inc}}))(L_{\tilde{J}} \cap M_{\tilde{K}}) \quad (1)$$

whenever $|\tilde{J}| = |J|$, $|\tilde{K}| = |K|$ and geometrically $L_{\tilde{J}} \cap M_{\tilde{K}} = L_J \cap M_K$ after forgetting the order. Here J^{inc} is the rearrangement of J in the increasing order. It is straightforward to see that

$$\mathcal{S}_{i,d} = \bigoplus'_{L' \in \mathcal{L}_{(i-d)/2}, M' \in \mathcal{M}_{n-(i+d)/2}} \mathcal{S}_i(L' \cap M') \quad (2)$$

is a finite-dimensional $\mathbb{Q}$ -vector space, where $\bigoplus'$ means that $\mathcal{S}_i(L' \cap M')$ is identified with $\mathcal{S}_i(\tilde{L}' \cap \tilde{M}')$ by (1).

PROPOSITION 3.2. *The space $\mathcal{S}_{i,d} = 0$ unless it lies in the above triangle bounded by $d = 0$, $i = d$ and $i + d = 2n$ and satisfies $i \equiv d \pmod{2}$.*

Proof. This is clear. □

REMARKS 3.3. To understand the diagram in Figure 1 geometrically we observe the following.

(i) The d th row of the diagram is constructed only from d -dimensional linear subspaces of $\mathbb{P}^n$.

(ii) In general, the further left or right we move, the more contributions come from M -faces or L -faces respectively. For example, $\mathcal{S}_{0,0} = \langle (M\text{-vertices}) \rangle$ and $\mathcal{S}_{2n,0} = \langle (L\text{-vertices}) \rangle$.

(B) For each projective subspace N' of $\mathbb{P}^n$, with N a codimension 1 subspace of N' , and each integer i we now define linear maps

$$u(N, N') : \mathcal{S}_i(N) \longrightarrow \mathcal{S}_{i-1}(N')$$

and

$$v(N', N) : \mathcal{S}_{i+1}(N') \longrightarrow \mathcal{S}_i(N).$$

For $j + k \leq n$ and $j, k \geq 0$ let

$$J = (\alpha_1, \dots, \alpha_j), \quad K = (\beta_1, \dots, \beta_k) \quad \text{and} \quad L_J \cap M_K \in \mathcal{L}_j\mathcal{M}_k.$$

Let $J' = J \setminus \{\alpha_j\}$ and $K' = K \setminus \{\beta_k\}$. By the structure of $\mathcal{S}_i$ we only need to

define the following. Put $i = n + j - k$:

$$\begin{aligned} u(L_J \cap M_K, L_J \cap M_{K'}) &= 0; & v(L_{J'} \cap M_K, L_J \cap M_K) &= 0; \\ u(L_J \cap M_K, L_{J'} \cap M_K) &: \mathcal{S}_i(L_J \cap M_K) \longrightarrow \mathcal{S}_{i-1}(L_{J'} \cap M_K), \\ &(L_J \cap M_K) \longmapsto (-1)^k(L_{J'} \cap M_K); \\ v(L_J \cap M_{K'}, L_J \cap M_K) &: \mathcal{S}_{i+1}(L_J \cap M_{K'}) \longrightarrow \mathcal{S}_i(L_J \cap M_K), \\ &(L_J \cap M_{K'}) \longmapsto (L_J \cap M_K). \end{aligned}$$

REMARKS 3.4. (i) The linear maps u and v are well defined. We only need to observe that if $L_J \cap M_{K'} = L_{J'} \cap M_K$ then $L_J \cap M_{K'} \subset M_{\beta_k}$ and therefore $\text{codim}(L_J \cap M_K) = \text{codim}(L_J \cap M_{K'}) < j + k$ which is contradictory to our choice of L_J and M_K .

(ii) If the element in $J \setminus J'$ is not the last entry in J then we can reorder J and make it happen. The same applies for K' .

PROPOSITION 3.5. *The linear maps u and v satisfy the following conditions, whenever the compositions named make sense:*

- (1a) $u(N, N') \circ v(N', N) = 0$,
- (1b) $v(N', N) \circ u(N, N') = 0$,
- (2a) for fixed N and N'' , $\sum_{N \subset N' \subset N''} v(N', N) \circ v(N'', N') = 0$,
- (2b) for fixed N and N'' , $\sum_{N \subset N' \subset N''} u(N', N'') \circ u(N, N') = 0$,
- (2c) for fixed N and N'' , $v(N'', N'_2) \circ u(N'_1, N'') + u(N, N'_2) \circ v(N'_1, N) = 0$.

Proof. The conditions (1a) and (1b) are easily checked by definition and Remark 3.4. To verify the rest we let $L_J, L_{J'}, M_K$ and $M_{K'}$ be as above and let $J'' = J' \setminus \{\alpha_{j-1}\}$ and $K'' = K' \setminus \{\beta_{k-1}\}$.

(2a) Let $(L_J; M_K) \in \text{GP}(L; M)$ and $N = L_J \cap M_K$. Cases where $N'' = L_{J''} \cap M_K$ or $N'' = L_{J'} \cap M_{K'}$ are trivial by definition. Assume $N'' = L_J \cap M_{K''}$. Then we have only two possible choices of N' :

- (i) $N'_1 = L_J \cap M_{K'}$ or
- (ii) $N'_2 = L_J \cap M_{\tilde{K}'}$ where $\tilde{K}' = (\beta_1, \dots, \beta_{k-2}, \beta_k)$.

In Case (i) we have

$$v(N'_1, N) \circ v(N'', N'_1)(L_J \cap M_{K''}) = v(N'_1, N)(L_J \cap M_{K'}) = (L_J \cap M_K),$$

whereas in Case (ii),

$$v(N'_2, N) \circ v(N'', N'_2)(L_J \cap M_{K''}) = v(N'_2, N)(L_J \cap M_{\tilde{K}'}) = -(L_J \cap M_K),$$

because $M_K = -M_{\beta_1, \dots, \beta_{k-2}, \beta_k, \beta_{k-1}}$. So (2a) is proved.

(2b) This is similar to (2a).

(2c) Let $(L_J; M_K) \in \text{GP}(L; M)$ and $N = L_J \cap M_K$. Cases where $N'' = L_{J''} \cap M_K$ or $N'' = L_J \cap M_{K''}$ are trivial by definition. Assume $N'' = L_{J'} \cap M_{K'}$. Then we have only two possible choices of N' :

- (i) $N'_1 = L_{J'} \cap M_K$ and $N'_2 = L_J \cap M_{K'}$; or
- (ii) $N'_1 = L_J \cap M_{K'}$ and $N'_2 = L_{J'} \cap M_K$.

Case (i) is trivial to verify. In Case (ii) we have

$$v(N'', N'_2) \circ u(N'_1, N'')(N'_1) = v(N'', N'_2)((-1)^{k-1}N'') = (-1)^{k-1}N'_2,$$

and

$$u(N, N'_2) \circ v(N'_1, N)(N'_1) = u(N, N'_2)(N) = (-1)^k N'_2.$$

Thus (2c) is verified in this case too. $\square$

REMARK 3.6. There are some misprints on page 705 of [3] in condition (2c). Pictorially, condition (2c) says that the following diagram anti-commutes:

$$\begin{array}{ccc} & N'' & \\ v \swarrow & & \nwarrow u \\ N'_2 & & N'_1 \\ u \swarrow & & \nwarrow v \\ & N & \end{array}$$

DEFINITION 3.7. For $\mathcal{S}_{i,d}$ in the triangular region of Figure 1 let $j = \frac{1}{2}(i - d)$ and $k = n - \frac{1}{2}(i + d)$ with $i \equiv d \pmod{2}$. We define

$$\begin{aligned} u : \mathcal{S}_{i,d} &\longrightarrow \mathcal{S}_{i-1,d+1}, \\ N &\longmapsto (-1)^k \sum'_{N' \in \mathcal{L}_{j-1} \mathcal{M}_k, N \subset N'} N'; \\ v : \mathcal{S}_{i,d} &\longrightarrow \mathcal{S}_{i-1,d-1}, \\ N &\longmapsto \sum'_{N' \in \mathcal{L}_j \mathcal{M}_{k+1}, N' \subset N} N'. \end{aligned}$$

Here $\sum'$ means that N' ranges over distinct linear subspaces of $\mathbb{P}^n$.

Geometrically, the map u sends a linear subspace produced by faces of L and M to all the subspaces of one dimension higher by removing one (highest dimensional) L -face each time; while the map v sends such a linear subspace to all the subspaces of one dimension lower by intersecting it with one (highest dimensional) M -face each time.

By definition we see that the 2-dimensional diagram $\mathcal{S}(L; M)$ satisfies the condition of a double complex except at terms on the bottom line. This follows from Proposition 3.5 and the fact that u is a finite sum of $u(N, N')$'s, and v is a finite sum of $v(N', N)$'s. Moreover, we can put a weight filtration $W_\bullet$ on the diagram by setting W_{2k} of the diagram to be that part of it lying on or to the lower left of those terms $\mathcal{S}_{i,d}$ with $i + d = 2k$ for $0 \leq k \leq n$ (compare [3, § 2.2.1]). Then we define $W_{2k+1} = W_{2k}$ for $0 \leq k < n$, $W_i = 0$ for $i < 0$, and $W_i = W_{2n}$ for $i > 2n$. Thus the weight graded piece gr_i^W is trivial unless $i = 0, 2, \dots, 2n - 2, 2n$. Now according to [3, § 2.2.3] we make the following definition.

DEFINITION 3.8. For any integer $0 \leq j \leq n$ we construct a cochain complex $\mathbf{K}_{2j}^\bullet(L; M)$ as follows. In the 2-dimensional diagram of Figure 1, we extract the parallelogram-shaped region with lower vertex at $\mathcal{S}_{2j,0}$ so that it becomes a double complex $\mathbf{K}_{2j}^{\bullet,\bullet}(L; M)$ such that

$$\mathbf{K}_{2j}^{p,q} = \mathcal{S}_{2j-p-q,p}.$$

Label the associated single complex $\mathbf{K}_{2j}^\bullet(L; M)$ so that the groups $\mathcal{S}_{2j,k}$ in the column over $\mathcal{S}_{2j,0}$ lie in degree zero.

Obviously, $\mathbf{K}_{2j}^q(L; M)$ is non-trivial if and only if $j - n \leq q \leq j$ because $\mathbf{K}_{2j}^{p,q} = 0$ unless (p, q) lies in the rectangular area of Figure 2.

$$\begin{array}{ccccccc}
 (j-n, j) & \bullet & \rightarrow & \bullet & \cdots & \rightarrow & \cdots & \bullet & \rightarrow & \bullet & (0, j) \\
 & \vdots & & \vdots & & & & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & \\
 & \vdots & & \vdots & & & & \vdots & & \vdots & \\
 (j-n, 0) & \bullet & \rightarrow & \bullet & \cdots & \rightarrow & \cdots & \bullet & \rightarrow & \bullet & (0, 0)
 \end{array}$$

FIGURE 2. The non-trivial terms of $\mathbf{K}_{2j}^{\bullet, \bullet}$, for $0 \leq j \leq n$.

We are now ready to define the motivic cohomology of a pair of admissible simplices.

DEFINITION 3.9. Let $(L; M)$ be a non-degenerate admissible pair of simplices in $\mathbb{P}_F^n$. Then we define the q th rational motivic cohomology of $(L; M)$ by

$$H_{\text{mot}}^q(L; M) := \bigoplus_{j=0}^n H_{\text{mot}}^q(L; M)_j, \quad H_{\text{mot}}^q(L; M)_j := H^q(\mathbf{K}_{2j}^\bullet(L; M)), \quad (3)$$

graded by the weights $2j$.

From the general theory of framed Hodge–Tate structures of §2.2 we expect that the weight filtration $W_\bullet$ makes the motivic cohomology $H_{\text{mot}}^*(L; M)$ into a graded comodule $\bigoplus_{j=0}^n H_{\text{mot}}^*(L; M)_j$ over the graded (commutative) Hopf algebra $A_{\bullet, \mathbb{Q}}$ (see equation (23)).

3.2. The motivation

This subsection is logically dependent on Main Theorem 3.25 concerning the structure of $H_{\text{mot}}^*(L; M)$. Readers are recommended to skip this section in the first reading. The goal of the present subsection is to motivate our construction in the previous subsection by proving the following theorem.

THEOREM 3.10 (compare [3, Theorem 2.2.3]). *Let $F = \mathbb{C}$ and $\mathbb{P}^n = \mathbb{P}_{\mathbb{C}}^n$. Let $(L; M)$ be an admissible pair of simplices in $\mathbb{P}^n$ in general position. Then the relative Betti cohomology*

$$H^{n+q}(\mathbb{P}^n \setminus L, M \setminus L; \mathbb{Q}) \cong \bigoplus_{j=0}^n H^q(\mathbf{K}_{2j}^\bullet(L; M))$$

graded by the weight.

Proof. For any complex projective variety X and any closed subset N of X let $j^U : U = X \setminus N \hookrightarrow X$ and $i^{N, X} : N \hookrightarrow X$ be the inclusions. Recall that if $\mathcal{F}$ is a

constant sheaf on an irreducible X then we have the exact sequence

$$0 \longrightarrow \mathcal{H}_N^0(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow j_*(\mathcal{F}|_U) \longrightarrow 0, \quad (4)$$

where $\mathcal{H}_N^0(X, \mathcal{F})$ is the subsheaf of $\mathcal{F}$ with supports in N (see [12, Exercise II.1.20]).

We use the same $\mathbb{Z}$ to denote the constant sheaf of $\mathbb{Z}$ on $\mathbb{P}^n$. As usual, we have a long exact sequence for Betti cohomology

$$\begin{aligned} \dots \longrightarrow H^{q-1}(M \setminus L, \mathbb{Z}) \longrightarrow H^q(\mathbb{P}^n \setminus L, M \setminus L; \mathbb{Z}) \longrightarrow \\ \longrightarrow H^q(\mathbb{P}^n \setminus L, \mathbb{Z}) \xrightarrow{\text{res}} H^q(M \setminus L, \mathbb{Z}) \longrightarrow \dots \end{aligned}$$

where res is the restriction map. Hence the cohomology groups $H^*(M \setminus L, \mathbb{Z})$ and $H^*(\mathbb{P}^n \setminus L, \mathbb{Z})$ are linked together and yield a two-column ($p = 0, 1$) first quadrant spectral sequence converging to the relative cohomology

$$E_1^{p,q} = H^q(M_{(p)} \setminus L, \mathbb{Z}) \implies H^{p+q}(\mathbb{P}^n \setminus L, M \setminus L; \mathbb{Z}), \quad (5)$$

where $M_{(0)} = \mathbb{P}^n$ and $M_{(1)} = M$.

To decode $H^p(M \setminus L, \mathbb{Z})$ we let $T_i = M_i \setminus L$ and for any subset

$$I = \{i_1, \dots, i_s\} \subset [n]$$

we put $T_I = \bigcap_j T_{i_j} = M_I \setminus L$. Set $\mathbf{M}_I \mathbf{L}_J = \mathcal{H}_{M_I L_J}^0(\mathbb{Z}|_{M_I})$ for any $I, J \subset [n]$, where if $I = \emptyset$ then $M_I = L_I = \mathbb{P}^n$. Then since M_I is irreducible in the Zariski topology, it can be proved easily by using (4) that we have the projective resolutions

$$0 \longrightarrow \bigoplus_{|J|=n} \mathbf{L}_J \xrightarrow{\partial} \dots \xrightarrow{\partial} \bigoplus_{|J|=1} \mathbf{L}_J \xrightarrow{\partial} \mathbb{Z} \xrightarrow{\partial} j_*^{\mathbb{P}^n \setminus L}(\mathbb{Z}|_{\mathbb{P}^n \setminus L}) \longrightarrow 0$$

and

$$0 \longrightarrow \bigoplus_{|J|=n-|I|} \mathbf{M}_I \mathbf{L}_J \xrightarrow{\partial} \dots \xrightarrow{\partial} \bigoplus_{|J|=1} \mathbf{M}_I \mathbf{L}_J \xrightarrow{\partial} \mathbf{M}_I \xrightarrow{\partial} j_*^{T_I}(\mathbb{Z}|_{T_I}) \longrightarrow 0.$$

Here the differentials ∂ are induced by the maps u from the previous subsection. Hence we can compute the following cohomology groups by these resolutions:

$$H^p(\mathbb{P}^n \setminus L, \mathbb{Z}) \cong R^p \Gamma(\mathbb{P}^n, j_*^{\mathbb{P}^n \setminus L}(\mathbb{Z}|_{\mathbb{P}^n \setminus L})), \quad (6)$$

$$H^p(M_I \setminus L, \mathbb{Z}) \cong R^p \Gamma(M_I, j_*^{M_I \setminus L}(\mathbb{Z}|_{M_I \setminus L})). \quad (7)$$

We now connect the cohomology $H^*(M \setminus L, \mathbb{Z})$ to $H^*(M_I \setminus L, \mathbb{Z})$ for I running through the non-trivial subsets of $[n]$. For any complex variety X covered by two open subsets U_1 and U_2 we have the Mayer–Vietoris long exact sequence

$$\begin{aligned} \dots \longrightarrow H^{q-1}(U_1 \cap U_2, \mathbb{Z}) \longrightarrow H^q(X, \mathbb{Z}) \longrightarrow \\ \longrightarrow H^q(U_1, \mathbb{Z}) \oplus H^q(U_2, \mathbb{Z}) \longrightarrow H^q(U_1 \cap U_2, \mathbb{Z}) \longrightarrow \dots \end{aligned}$$

which yields another two-column first quadrant spectral sequence ($p = 0, 1$)

$$E_1^{p,q} = \bigoplus_{I \subset \{1,2\}, |I|=p+1} H^q(U_I, \mathbb{Z}) \implies H^{p+q}(X, \mathbb{Z}),$$

where $U_{\{i\}} = U_i$ and $U_{\{1,2\}} = U_1 \cap U_2$. It is straightforward to extend this to the case where X is covered by finitely many open subsets and get

$$E_1^{p,q} = \bigoplus_{I \subset [n], |I|=p+1} H^q(T_I, \mathbb{Z}) \implies H^{p+q}(M \setminus L, \mathbb{Z}). \quad (8)$$

Here all the horizontal differentials $d_1^{p,q}$ are induced by the maps v from the previous subsection.

Splicing (5) and (8) together we get the first quadrant spectral sequence with

$$E_1^{p,q}(L, M) = \bigoplus_{I \subset [n], |I|=p} H^q(T_I, \mathbb{Z}) \implies H^{p+q}(\mathbb{P}^n \setminus L, M \setminus L; \mathbb{Z}).$$

Here if $|I| = 0$ then $T_I = \mathbb{P}^n \setminus L$. Notice that the horizontal maps from the first column to the second column are induced from the restriction maps $\mathbb{P}^n \rightarrow M_i$. Applying (6) and (7) we can define the E_0 -term of this spectral sequence as

$$E_0^{p,q}(L, M) = \bigoplus_{|I|=p, |J|=q} \Gamma(M_I, \mathbf{M}_I \mathbf{L}_J).$$

Let $B^{p,q}$ be the corresponding bicomplex whose single total complex is filtered by the columns

$$(\text{fil}_k(B^\bullet))^{p,q} = \begin{cases} B^{p,q} & \text{if } p \geq n - k, \\ 0 & \text{if } p < n - k. \end{cases}$$

We now put the increasing weight filtration W on B by letting $W_{2k} = \text{fil}_k$ and $W_{2k+1} = W_{2k}$ for all k . Define the bicomplex $A^{\bullet,\bullet}$ by $A^{p,q} = \mathcal{L}_{n-q} \mathcal{M}_p$ with differentials provided by the maps u and v from the previous subsection. It is clear that

$$E_2^{p,q}(L, M) \otimes \mathbb{Q} = H_h^p H_v^{n-q}(B^{\bullet,\bullet}) \otimes \mathbb{Q} \cong H_h^p H_v^q(A^{\bullet,\bullet}),$$

where H_h and H_v denote the horizontal and vertical cohomology, respectively. On the other hand,

$$E_2^{p,q}(L, M) \otimes \mathbb{Q} \cong \text{gr}_{2n-2p}^W H^{n+p-q}(\mathbb{P}^n \setminus L, M \setminus L; \mathbb{Q})$$

because the spectral sequence $E(L, M)$ degenerates at the E_2 -term since

$$H_h^p H_v^q(A^{\bullet,\bullet}) = \begin{cases} H^0(\mathbf{K}_{2n-2p}^{\bullet,\bullet}) & \text{if } p = q, \\ 0 & \text{otherwise.} \end{cases}$$

To see this consider the descriptive picture of $A^{\bullet,\bullet}$ given by Figure 3.

FIGURE 3. The bicomplex $A^{\bullet,\bullet}$.

Clearly, we may assume $0 \leq p \leq n$ and $p \leq q \leq n$. If $p < q$ then $A^{p,q}$ lies inside the bicomplex $\mathbf{K}_{2n-2p}^{\bullet,\bullet}$. Hence $H_v^q(A^{p,\bullet}) = H_v^{q-p}(\mathbf{K}_{2n-2p}^{0,\bullet})$ and $H_h^p H_v^q(A^{\bullet,\bullet})$ is a subquotient of

$$H_h^0 H_v^{q-p}(\mathbf{K}_{2n-2p}^{\bullet,\bullet}) \cong \text{gr}_{2n-2p}^W H_v^{q-p}(\mathbf{K}_{2n-2p}^{\bullet,\bullet}),$$

where the weight filtration on $\mathbf{K}$ is inherited from that of B . But $H^a(\mathbf{K}_{2n-2p}^{\bullet,\bullet}) = 0$

if $a > 0$ by Main Theorem 3.25. By the same theorem we know that $H^0(\mathbf{K}_{2n-2p}^\bullet)$ is of pure weight $2n - 2p$ and therefore if $p = q$ then

$$H_h^p H_v^p(A^{\bullet,\bullet}) = H_h^0 H_v^0(\mathbf{K}_{2n-2p}^{\bullet,\bullet}) = H^0(\mathbf{K}_{2n-2p}^\bullet).$$

This proves the required result and concludes the proof of our theorem. $\square$

The above theorem is the motivation of Definition 3.9 in the previous subsection. From the proof we see that the space $H^n(\mathbb{P}^n \setminus L, M \setminus L; \mathbb{Z})$ has a canonical mixed Hodge structure [7, (8.3.10)] which we denote by $H(L; M)$. The weights of $H(L; M)$ that are of interest are $2k$ for $0 \leq k \leq n$ and the corresponding graded quotients are pure Hodge–Tate structures isomorphic to $\mathbb{Z}(-k)^{c_k}$ which is c_k copies of the Tate structure of weight $2k$ for some non-negative integer c_k . Now consider the complex $(E_1^{p,0} \otimes \mathbb{Q}, d_1^{p,0})$. It is nothing but the injective resolution of $j_!^{\mathbb{P}^n \setminus M}(\mathbf{Q}|_{\mathbb{P}^n \setminus M})$ where $\mathbf{Q}$ is the constant sheaf of $\mathbb{Q}$ on $\mathbb{P}^n$. Therefore we see that

$$\mathrm{gr}_0^W H(L; M)_{\mathbb{Q}} \cong E_2^{n,0} \otimes \mathbb{Q} = H_{\mathrm{cpt}}^n(\mathbb{P}^n \setminus M, \mathbb{Q})$$

which is the compactly supported cohomology of $\mathbb{P}^n \setminus M$ and therefore is dual to the Borel–Moore homology $H_n^{\mathrm{B.M.}}(\mathbb{P}^n \setminus M; \mathbb{Q})$. On the other hand,

$$\mathrm{gr}_{2n}^W H(L; M) \cong E_2^{0,n} \cong E_1^{0,n} = H^n(\mathbb{P}^n \setminus L, \mathbb{Z}).$$

Therefore the mixed Hodge structure $H(L; M)$ has the following frame: a distinguished covector

$$\Delta_M : \mathrm{gr}_0^W H(L; M) \simeq \mathbb{Z}(0),$$

where Δ_M is the class of the oriented chain representing a generator of the Borel–Moore homology $H_n^{\mathrm{B.M.}}(\mathbb{P}^n \setminus M; \mathbb{Z})$; and a distinguished vector

$$[\omega_L] : \mathbb{Z}(-n) \simeq \mathrm{gr}_{2n}^W H(L; M),$$

where $[\omega_L]$ is the class of

$$(2\pi i)^{-n} d \log(t_1/t_0) \wedge \dots \wedge d \log(t_n/t_0), \quad L_i = \{t_i = 0\}, \quad \text{for } 0 \leq i \leq n.$$

Observe that $(2\pi i)^n [\omega_L]$ lies in $H^n(\mathbb{P}^n \setminus L, \mathbb{Z})$ under the isomorphism (cf. [6, (3.2.2)])

$$H^n(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^\bullet(\log L)) \cong H^n(\mathbb{P}^n \setminus L, \mathbb{C}).$$

Finally, we notice that the Aomoto polylogarithm $\int_{\Delta_M} \omega_L$ satisfies similar axioms to those in the definition of A_n (Definition 2.1). Thus the map $(L; M) \mapsto H(L; M)$ providing the homomorphism $A_n \rightarrow \mathcal{H}_n$ is a formal analog of the Aomoto polylogarithm.

3.3. A pairing

As observed in [2] we can define the pairings on the $\mathbb{Q}$ -vector spaces

$$\mathbf{K}_{2n-2i}^q(L; M) \times \mathbf{K}_{2i}^{-q}(M; L) \xrightarrow{\langle\langle \cdot, \cdot \rangle\rangle} \mathbb{Q}$$

for $-i \leq q \leq n - i$ as follows: if x and y can be represented by the same $(L_J \cap M_K) \in \mathcal{L}_j \mathcal{M}_k$ then

$$\langle\langle x, y \rangle\rangle = (-1)^{jk}.$$

For all other x and y we define $\langle\langle x, y \rangle\rangle = 0$. Then we linearly extend $\langle\langle \cdot, \cdot \rangle\rangle$ to $\mathbf{K}_{2n-2i}^q(L; M) \times \mathbf{K}_{2i}^{-q}(M; L)$.

PROPOSITION 3.11. *The pairing $\langle\langle \cdot, \cdot \rangle\rangle$ provides an isomorphism between the two complexes of $\mathbb{Q}$ -vector spaces*

$$\mathbf{K}_{2n-2i}(L; M) \cong \mathbf{K}_{2i}(M; L)^*.$$

Here the dual is obtained by applying the contravariant functor $\text{Hom}_{\mathbb{Q}}(-, \mathbb{Q})$.

Proof. We only need to verify that the pairings are compatible with the differentials. Let u^* and v^* be the differentials for $\mathbf{K}_{2i}(M; L)^*$. Suppose $(L_J \cap M_K) \in \mathcal{L}_j \mathcal{M}_k$ with $J = \{\alpha_1, \dots, \alpha_j\}$ and $K = \{\beta_1, \dots, \beta_k\}$. Let $J' = J \setminus \{\alpha_j\}$ and $K' = K \setminus \{\beta_k\}$. Then we have

$$\langle\langle u(N, N')(L_J \cap M_K), (L_{J'} \cap M_{K'}) \rangle\rangle = 0,$$

$$\langle\langle (L_J \cap M_K), v^*(N', N)(L_{J'} \cap M_{K'}) \rangle\rangle = 0,$$

unless $N = (L_J \cap M_K)$ and $N' = (L_{J'} \cap M_{K'})$ in which case we have

$$\langle\langle u(N, N')(L_J \cap M_K), (L_{J'} \cap M_{K'}) \rangle\rangle = (-1)^{jk},$$

$$\langle\langle (L_J \cap M_K), v^*(N', N)(L_{J'} \cap M_{K'}) \rangle\rangle = (-1)^{jk}.$$

Similarly, one can prove that

$$\langle\langle (L_J \cap M_K), v(N', N)(L_J \cap M_{K'}) \rangle\rangle = \langle\langle u^*(N, N')(L_J \cap M_K), (L_J \cap M_{K'}) \rangle\rangle$$

which is either $(-1)^{jk}$ if $N = (L_J \cap M_K)$ and $N' = (L_J \cap M_{K'})$ or 0 otherwise.

The proposition is now proved. $\square$

Applying Proposition 3.11 we readily get the following corollary.

COROLLARY 3.12. *Let $(L; M)$ be an admissible pair of simplices. For $0 \leq j \leq n$ we set $G_j(L; M) = H^0(\mathbf{K}_{2j}^\bullet(L; M))$. Then there is an isomorphism*

$$G_j(L; M) \cong G_{n-j}(M; L)^*.$$

REMARKS 3.13. (i) One can imagine the isomorphism pictorially as a flipping of the 2-dimensional diagram in Figure 1 about the center column (that is, the n th column) followed by reversing all the arrows and changing u to v^* and v to u^* .

(ii) Our definition of the complex $\mathbf{K}_{2j}(L; M)$ and the above pairing does not agree with [2] when $n = 2$. But it is obvious that up to signs they are the same. We make these changes for the sake of easy generalization.

3.4. Construction of the complex $C(L; M)$

In this subsection we will recast the objects $\mathcal{S}_{i,d}$ as abelian groups $D_{(i-d)/2, n-(i+d)/2}$ defined by generators and relations and define differentials d' and d'' corresponding to u and v respectively. They provide an integral version of $\mathcal{S}(L; M)$ and, more importantly, are very convenient for computation.

Recall that for a subset $\{\beta_1, \dots, \beta_k\}$ of $\llbracket n \rrbracket$ we set $M_{\beta_1, \dots, \beta_k} = (M_{\beta_1}, \dots, M_{\beta_k})$ which is an M -face with codimension k (geometrically, it is $\bigcap_{s=1}^k M_{\beta_s}$). We call a

chain of inclusions of the faces

$$M_{\beta_1, \dots, \beta_n} \subset M_{\beta_1, \dots, \beta_{n-1}} \subset \dots \subset M_{\beta_1, \beta_2} \subset M_{\beta_1}$$

an M -flag, denoted by $M[\beta_1, \dots, \beta_k]$ or $(M_{\beta_1, \dots, \beta_k}, \dots, M_{\beta_1})$. There are $(n+1)!$ different M -flags. Partial flags are in one-to-one correspondence with the faces of M by

$$M[\beta_1, \dots, \beta_k] = (M_{\beta_1, \dots, \beta_k}, \dots, M_{\beta_1}) \longleftrightarrow M_{\beta_1, \dots, \beta_k}.$$

We define L -flags similarly.

DEFINITION 3.14. Set $D_{j,k} = 0$ if $j+k \geq n+1$ or $j < 0$ or $k < 0$. Set $D_{0,0} = \langle (\mathbb{P}^n) \rangle$. For $k \neq 0$ or $j \neq 0$, $D_{j,k} = \text{Gen}(j,k)/\text{Rel}(j,k)$ where $\text{Gen}(j,k)$ is the group generated by finite $\mathbb{Z}$ -linear combinations of pairs of flags:

$$\sum_{\substack{L^j \subset \dots \subset L^1, L^s \in \mathcal{L}_s \\ M^k \subset \dots \subset M^1, M^t \in \mathcal{M}_t}} a_{(L^j, \dots, L^1; M^k, \dots, M^1)} (L^j, \dots, L^1; M^k, \dots, M^1). \quad (9)$$

Here the coefficients $a_{(L^j, \dots, L^1; M^k, \dots, M^1)} \in \mathbb{Z}$ should satisfy two kinds of conditions.

(i) For any fixed $l < j$, fixed $L^s \in \mathcal{L}_s$ (with $s \neq l$), fixed $M^t \in \mathcal{M}_t$ (with $1 \leq t \leq k$), and variable $L^l \in \mathcal{L}_l$,

$$\sum_{L^{l+1} \subset L^l \subset L^{l-1}} a_{(L^j, \dots, L^1; M^k, \dots, M^1)} = 0. \quad (10)$$

(ii) If $L^j \cap \tilde{M}^t = L^j \cap M^t$ for all $1 \leq t \leq k$ then

$$a_{(L^j, \dots, L^1; M^k, \dots, M^1)} = a_{(L^j, \dots, L^1; \tilde{M}^k, \dots, \tilde{M}^1)}. \quad (11)$$

The group $\text{Rel}(j,k)$ is generated by two kinds of relations.

(i) For every fixed $l < k$, every fixed $M^t \in \mathcal{M}_t$ (with $t \neq l$), every fixed $L^s \in \mathcal{L}_s$ (with $1 \leq s \leq j$), and variable $M^l \in \mathcal{M}_l$,

$$\sum_{M^{l+1} \subset M^l \subset M^{l-1}} (L^j, \dots, L^1; M^k, \dots, M^1) = 0. \quad (12)$$

(ii) If $\tilde{L}^s \cap M^k = L^s \cap M^k$ for all $1 \leq s \leq j$ then

$$(L^j, \dots, L^1; M^k, \dots, M^1) = (\tilde{L}^j, \dots, \tilde{L}^1; M^k, \dots, M^1). \quad (13)$$

The differentials d' and d'' are defined as follows:

$$\begin{aligned} d &= d_{j,k} = d'' + d' : \\ D_{j,k} &\longrightarrow D_{j,k+1} \oplus D_{j-1,k}, \\ (L^j, \dots, L^1; M^k, \dots, M^1) &\longmapsto \sum_{\substack{M' \in \mathcal{M}_{k+1}, M' \subset M^k \\ M' \cap (L^j, \dots, L^1) \text{ distinct}}} (L^j, \dots, L^1; M', M^k, \dots, M^1) \\ &\quad + \frac{(-1)^k}{\#(\Sigma)} \sum_{(\tilde{M}^k, \dots, \tilde{M}^1) \cap L^j = (M^k, \dots, M^1) \cap L^j} (L^{j-1}, \dots, L^1; \tilde{M}^k, \dots, \tilde{M}^1), \end{aligned}$$

where $\#(\Sigma)$ is the number of terms in the sum. Then d is linearly extended to $D_{j,k}$.

Here if $j = 0$ then $d' = 0$ and if $j + k = n$ then $d'' = 0$. By convention we set

$$\begin{aligned} d: D_{1,0} &\longrightarrow D_{0,0} \oplus D_{1,1}, \\ (L^1) &\longmapsto (\mathbb{P}^n) + \sum_{M^1 \in \mathcal{M}_1} (L^1; M^1), \\ d: D_{0,0} &\longrightarrow D_{0,1}, \\ (\mathbb{P}^n) &\longmapsto \sum_{M^1 \in \mathcal{M}_1} (M^1). \end{aligned}$$

REMARKS 3.15. (i) It is not hard to see that the fraction appearing in the above definition is superficial. Indeed, if the flag $(L^j, \dots, L^1; M^k, \dots, M^1)$ appears in $x \in D(j, k)$ with coefficient a , then all the flags $(L^j, \dots, L^1; \tilde{M}^k, \dots, \tilde{M}^1)$, with $L^j \cap \tilde{M}^t = L^j \cap M^t$ for all $1 \leq t \leq k$, must appear in x with the same coefficient a by (11). Hence every term of $d(x)$ has integer coefficient.

(ii) In the definition of d'' we only allow distinct $M' \cap (L^j, \dots, L^1)$ to appear because ultimately we will be concerned only with the geometric configuration of $(L; M)$.

(iii) The key idea is that we should keep skew-symmetry on both L and M by (10) and (12) (see Lemma 3.18). The relations (11) and (13) take the non-generic conditions into account.

In the next definition we construct the complex $C(L; M)$ using groups $D_{j,k}$ as building blocks.

DEFINITION 3.16. Let $1 \leq j \leq n$. We define the complex $C_{2j}^\bullet(L; M)$ as follows: for $j - n \leq k \leq j$,

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_{2j}^{2j-n-k} & \xrightarrow{d_{2j}^{2j-n-k}} & C_{2j}^{2j-n-k+1} & \longrightarrow & \cdots \\ & & \parallel & & \parallel & & \\ \cdots & \longrightarrow & \bigoplus_{r=r_1(k)}^{r_2(k)} D_{r,r-k} & \xrightarrow{d} & \bigoplus_{r=r_1(k-1)}^{r_2(k-1)} D_{r,r-k+1} & \longrightarrow & \cdots \end{array}$$

where if $2j - n \geq 0$ then

$$(r_1(k), r_2(k)) = \begin{cases} (0, n + k - j) & \text{if } j - n \leq k \leq 0, \\ (k, n + k - j) & \text{if } 0 \leq k \leq 2j - n, \\ (k, j) & \text{if } 2j - n \leq k \leq j, \end{cases}$$

and if $2j - n \leq 0$ then

$$(r_1(k), r_2(k)) = \begin{cases} (0, n + k - j) & \text{if } j - n \leq k \leq 2j - n, \\ (0, j) & \text{if } 2j - n \leq k \leq 0, \\ (k, j) & \text{if } 0 \leq k \leq j. \end{cases}$$

Here we use the fact that if $r \geq n + k - j$ and $j - n \leq k$ then $2r - k + 1 \geq n + 1$ and therefore $D_{r,r-k+1} = 0$.

LEMMA 3.17. *Let $(L; M)$ be an admissible pair of simplices. Then for all j and k we can define $D_{j,k}$ by using only pairs of sub-simplices in $\text{GP}(L; M)$.*

Proof. We want to show that $D_{j,k}$ can actually be generated by sums as in (9) where equations (10) and (11) hold and, moreover, $(L^j; M^k) \in \text{GP}(L; M)$ whenever $a_{(L^j, \dots, L^1; M^k, \dots, M^1)} \neq 0$. Suppose there appears a pair $(L^j; M^k) \notin \text{GP}(L; M)$. Then $\text{codim}(L^j \cap M^k) < j + k$ and $j, k \geq 1$. Write the flag

$$(L^j, \dots, L^1; M^k, \dots, M^1) = (L[\alpha_1, \dots, \alpha_j]; M[\beta_1, \dots, \beta_k]).$$

We now have two cases.

Case (i): $L^j \not\subset M^1 = M_{\beta_1}$. Then we must have some $l < k$ such that

$$M_{\beta_1 \dots \beta_l} \cap L^j = M_{\beta_1 \dots \beta_{l-1} \beta_{l+1}} \cap L^j. \quad (14)$$

Indeed, if $M_{\beta_1 \dots \beta_l} \cap L^j \not\subset M_{\beta_{l+1}}$ for all $1 \leq l < k$ then we see that

$$\text{codim}(M_{\beta_1 \dots \beta_{l+1}} \cap L^j) = \text{codim}(M_{\beta_1 \dots \beta_l} \cap L^j) + 1, \quad \text{for } 1 \leq l < k.$$

Thus

$$\text{codim}(M_{\beta_1 \dots \beta_k} \cap L^j) = \text{codim}(M^1 \cap L^j) + k - 1 = j + k$$

which is a contradiction. By equation (12) we get

$$(L^j, \dots, L^1; M[\beta_1, \dots, \beta_k]) + (L^j, \dots, L^1; M[\beta_1, \dots, \beta_{l-1}, \beta_{l+1}, \beta_l, \beta_k]) = 0,$$

while by equations (14) and (11) the coefficients of these two terms are equal in the expression (9). Thus the flag $(L^j, \dots, L^1; M[\beta_1, \dots, \beta_k])$ cannot essentially appear in (9).

Case (ii): $L^j \subset M^1$. Then

$$L_{\alpha_2 \alpha_3 \dots \alpha_j} \cap M^1 = L_{\alpha_1 \alpha_3 \dots \alpha_j} \cap M^1.$$

By (13) we have

$$(L[\alpha_1, \dots, \alpha_j]; M[\beta_1, \dots, \beta_k]) = (L[\alpha_2, \alpha_1, \alpha_3, \dots, \alpha_j]; M[\beta_1, \dots, \beta_k]).$$

But by (11),

$$a_{(L[\alpha_1, \dots, \alpha_j]; M[\beta_1, \dots, \beta_k])} + a_{(L[\alpha_2, \alpha_1, \alpha_3, \dots, \alpha_j]; M[\beta_1, \dots, \beta_k])} = 0.$$

Hence again the flag $(L^j, \dots, L^1; M[\beta_1, \dots, \beta_k])$ cannot essentially appear in (9). $\square$

LEMMA 3.18. *Every element in $D_{j,k}$ satisfies skew-symmetry on both its L -part and its M -part. Namely, for any permutations σ of $\{1, \dots, j\}$ and τ of $\{1, \dots, k\}$,*

$$\begin{aligned} & (L[\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(j)}]; M[\beta_{\tau(1)}, \dots, \beta_{\tau(k)}]) \\ &= \text{sgn}(\sigma) \text{sgn}(\tau) (L[\alpha_1, \dots, \alpha_j]; M[\beta_1, \dots, \beta_k]). \end{aligned}$$

Proof. First we deal with the M -part. By definition, for any $\beta_1, \dots, \beta_k \in \llbracket n \rrbracket$, any fixed $L^s \in \mathcal{L}_s$ (with $1 \leq s \leq j$), any fixed $M^t = M_{\beta_1 \dots \beta_t} \in \mathcal{M}_t$ (with $1 \leq t \leq k$),

$t \neq l$), and variable $M^l \in \mathcal{M}_l$,

$$\begin{aligned} 0 &= \sum_{M^{l+1} \subset M^l \subset M^{l-1}} (L^j, \dots, L^1; M^k, \dots, M^1) \\ &= (L^i, \dots, L^1; M[\beta_1, \dots, \beta_k]) + (L^i, \dots, L^1; M[\beta_1, \dots, \beta_{l+1}, \beta_l, \beta_{l+2}, \dots, \beta_k]), \end{aligned}$$

that is, $D(j, k)$ satisfies skew-symmetry on the part of M . In fact it is clear that skew-symmetry on the M -part is equivalent to (12). The assertion on the L -part can be verified using duality between L and M (see Proposition 3.11). Thus skew-symmetry on the L -part is equivalent to (10). $\square$

For example,

$$\begin{aligned} (M_{012}, M[01]) &= -(M_{012}, M[10]) = (M_{012}, M[12]) \\ &= -(M_{012}, M[21]) = (M_{012}, M[20]) = -(M_{012}, M[02]). \end{aligned}$$

Now suppose (L_{012}, L_{01}, L_0) has coefficient a , then (L_{012}, L_{01}, L_1) has coefficient $-a$, then (L_{012}, L_{12}, L_1) has coefficient a , and so on.

In order to compare $\mathbb{Q}$ -vector spaces $\mathcal{S}_{i,d}$ with D -groups we need to look at their bases. We first look for an optimal way to write down the basis of each of $\mathcal{S}_{i,d}$. For $(L_J; M_K) \in \text{GP}(L; M)$, let

$$\begin{aligned} \mathcal{E}(L_J; M_K) &= \{(L_{\tilde{J}}; M_{\tilde{K}}) \in \text{GP}(L; M) : \tilde{J} = \tilde{J}^{\text{inc}}, \tilde{K} = \tilde{K}^{\text{inc}}, |\tilde{J}| = |J|, \\ &\quad |\tilde{K}| = |K|, L_{\tilde{J}} \cap M_{\tilde{K}} = L_J \cap M_K\}. \end{aligned}$$

DEFINITION 3.19. For fixed $j, k > 0$ let us put the usual lexicographic order $\prec$ on $\{(I_1, I_2) : |I_1| = j, |I_2| = k, I_1, I_2 \subset \llbracket n \rrbracket\}$. We say that (J, K) is *smaller* than (J', K') if $(J, K) \prec (J', K')$. Then in the set $\mathcal{E}(L_J; M_K)$ there is a unique pair $(\tilde{J}, \tilde{K})$ which is smallest. We call the pair $(L_{\tilde{J}}; M_{\tilde{K}})$ of subsimplices of $(L; M)$ an *optimal pair*. Denote the set of all optimal pairs by $\text{OP}(L; M)$.

By formula (2) we get the following lemma.

LEMMA 3.20. *The finite-dimensional $\mathbb{Q}$ -vector space $\mathcal{S}_{i,d}$ has a basis*

$$\{(L_J \cap M_K) : (L_J; M_K) \in \text{OP}(L; M), 2|J| = 2n - d - i, 2|K| = i - d\}.$$

Proof. By definition, the increasing order of indices takes care of the skew-symmetry on both L and M ; ‘optimal’ restriction is aimed at the non-generic conditions. $\square$

REMARK 3.21. The optimality is a non-trivial condition. For example, take a non-degenerate admissible pair of tetrahedra $(L_0, \dots, L_3; M_0, \dots, M_3)$ in $\mathbb{P}^3$ with the only non-generic condition

$$L_{1,2} \cap M_{1,2} \neq \emptyset.$$

Then, in $\mathcal{S}_{2,0}$ we have

$$(L_1 \cap M_{1,2}) = (L_2 \cap M_{1,2}) = (L_1 \cap M_{2,1}) = (L_2 \cap M_{2,1}).$$

Among them only $(L_1 \cap M_{1,2}) \in \text{OP}(L; M)$.

LEMMA 3.22. *If $(L_J; M_K)$ is optimal then $(L_{\tilde{J}}; M_{\tilde{K}})$ is optimal for any subset $\tilde{J} \subset J$ and $\tilde{K} \subset K$ in the increasing order.*

Proof. We first show that the lemma is true if $\tilde{K} = K$ is fixed. The lemma follows from this because, by the same argument, we may fix $\tilde{J}$ and let $\tilde{K}$ vary.

Let $J = (\alpha_1, \dots, \alpha_j)$. Suppose on the contrary that there is an index subset $\tilde{J} \subset J$ such that $(L_{\tilde{J}}; M_K)$ is not optimal. Then there exists an optimal $(L_{J_1}; M_{K_1})$ with $J_1 = (r_1, \dots, r_p) \prec \tilde{J}$ and

$$L_{J_1} \cap M_{K_1} = L_{\tilde{J}} \cap M_K. \quad (15)$$

Step (i). We first show that $\tilde{J} \subset J_1$. Suppose on the contrary that $\tilde{J} \not\subset J_1$ and therefore there exists an $\alpha_s \in \tilde{J} \setminus J_1$. Then

$$L_{J_1} \cap M_{K_1} = L_{\tilde{J}} \cap M_K \subset L_{\alpha_s}$$

and thus

$$L_{J_1 \setminus \{r_p\}, \alpha_s} \cap M_{K_1} = L_{J_1} \cap M_{K_1}.$$

Hence $\alpha_s > r_p$ by optimality of $(L_{J_1}; M_{K_1})$. We claim that this in turn implies that $J_1 \subset J$. Suppose on the contrary that $r_i \notin J$ for some $1 \leq i \leq p$. Then

$$L_J \cap M_K \subset L_{\tilde{J}} \cap M_{\tilde{K}} \subset L_{r_i}.$$

Thus $L_J \cap M_K = L_{J_2} \cap M_K$ with

$$J_2 = (\alpha_1, \dots, \alpha_{s-1}, r_i, \alpha_{s+1}, \dots, \alpha_j) \prec J$$

because $r_i \leq r_p < \alpha_s$. This is contradictory to the assumption that $(L_J; M_K)$ is optimal. Hence $J_1 \subset J$.

To reach a contradiction under the assumption $\tilde{J} \not\subset J_1$ we now observe that because $J_1 \prec \tilde{J}$ we can reorder the index set $(J \setminus \tilde{J}) \cup J_1$ to get J_2 satisfying $J_2 \prec J$. But this is impossible because $(L_J; M_K)$ is optimal while

$$L_J \cap M_K = L_{J \setminus \tilde{J}} \cap (L_{\tilde{J}} \cap M_K) = L_{J \setminus \tilde{J}} \cap (L_{J_1} \cap M_{K_1}) = L_{J_2} \cap M_{K_1}.$$

Step (ii). Step (i) implies that $\tilde{J} = J_1$ because $J_1 \prec \tilde{J} \subset J_1$. We can now assume on the contrary that $(L_{\tilde{J}}; M_K)$ is not optimal and therefore $L_{\tilde{J}} \cap M_K = L_{\tilde{J}} \cap M_{K_1}$ with $K_1 \prec K$ by (15). Then

$$L_J \cap M_K = L_{J \setminus \tilde{J}} \cap (L_{\tilde{J}} \cap M_K) = L_{J \setminus \tilde{J}} \cap (L_{\tilde{J}} \cap M_{K_1}) = L_J \cap M_{K_1},$$

which contradicts the assumption that $(L_J; M_K)$ is optimal.

This completes the proof of the lemma. $\square$

Let us turn to the D -groups. In order to find bases of them we set $L(\alpha) = L_\alpha$ and $AL(\alpha) = \mathbb{P}^n$ for $0 \leq \alpha \leq n$. For $s \geq 1$ and $0 \leq \alpha_0 < \dots < \alpha_s \leq n$ we define

recursively

$$AL(\alpha_0, \dots, \alpha_s) = \sum_{i=0}^s (-1)^i L(\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_s), \quad \text{for } s \geq 1,$$

and

$$L(\alpha_0, \dots, \alpha_s) = (L_{\alpha_0 \dots \alpha_s}, AL(\alpha_0, \dots, \alpha_s)), \quad \text{for } s \geq 1.$$

LEMMA 3.23. *If $(L; M)$ is a generic pair of simplices then $D_{j,k}$ has the following basis:*

$$\{(L(\alpha_1, \dots, \alpha_j); M[\beta_1, \dots, \beta_k]) : 0 \leq \alpha_1 < \dots < \alpha_j \leq n, \\ 0 \leq \beta_1 < \dots < \beta_k \leq n\}. \quad (16)$$

Proof. By Lemma 3.18, each element in the set (16) is clearly the unique representation corresponding to the face $L_{\alpha_1, \dots, \alpha_j} \cap M_{\beta_1, \dots, \beta_k}$. The linear independence also follows easily. $\square$

To find a basis of $D_{j,k}$ in general, let us put

$$\mathcal{I} = \{(\beta_1, \dots, \beta_k) : 0 \leq \beta_1 < \dots < \beta_k \leq n\}.$$

For a finite set $B = \{b_i : i \in \mathcal{I}\}$ indexed by $\mathcal{I}$ and a partition $\mathcal{P} = \{\mathcal{I}_1, \dots, \mathcal{I}_s\}$ of $\mathcal{I}$ we define the *partition sum-set* of B corresponding to $\mathcal{P}$ as

$$B_{\mathcal{P}} = \left\{ e_r = \sum_{i \in \mathcal{I}_r} b_i : 1 \leq r \leq s \right\}.$$

Then by definition it is not difficult to see that there is always some partition sum-set $B_{\mathcal{P}}$ of (16), say,

$$B_{\mathcal{P}} = \left\{ e_r = \sum_{K \in \mathcal{I}_r} (L(J); M[K]) \right\},$$

that forms the basis of $D_{j,k}$. Here if both $(L(J); M[K_1])$ and $(L(J); M[K_2])$ appear in e_r then $L_J \cap M_{K_1} = L_J \cap M_{K_2}$. Let $(L(J); M[K])$ be any one of the terms appearing in e_r . Then there is a unique element

$$(J^{\text{OP}}, K^{\text{OP}}) \in \text{OP}(L; M) \cap \mathcal{E}(L(J); M[K])$$

so that we can rewrite

$$e_r = \sum_{K \in \mathcal{I}_r} (L(J^{\text{OP}}); M[K]) = e_r^{\text{OP}}.$$

Therefore $D_{j,k}$ can be generated by all such e^{OP} and they are linearly independent.

PROPOSITION 3.24. *For all j and k we have*

$$D_{j,k} \otimes \mathbb{Q} \cong \mathcal{S}_{n+j-k, n-j-k}. \quad (17)$$

Moreover, for each $i = 0, 1, \dots, n$, we have the isomorphism of complexes

$$C_{2i}(L; M) \otimes \mathbb{Q} \cong \mathbf{K}_{2i}(L; M).$$

Proof. Define a $\mathbb{Q}$ -linear map

$$\begin{aligned} \Phi' : D_{j,k} \otimes \mathbb{Q} &\longrightarrow \mathcal{S}_{n+j-k, n-j-k}, \\ (L(\alpha_1, \dots, \alpha_j); M[\beta_1, \dots, \beta_k]) &\longmapsto (L_{\alpha_1, \dots, \alpha_j} \cap M_{\beta_1, \dots, \beta_k}). \end{aligned}$$

Then we extend it linearly to $D_{j,k}$. It is easy to see that

$$\Phi'(e_r^{\text{OP}}) = |\mathcal{I}_r|(L_{J^{\text{OP}}}; M_{K^{\text{OP}}}).$$

Setting $\Phi(e_r^{\text{OP}}) = \Phi'(e_r^{\text{OP}})/|\mathcal{I}_r|$ proves (17) immediately thanks to Lemma 3.20.

We now show that the differentials are compatible with Φ . Using the above notation we put

$$x = e_r \in D_{j,k}, \quad x_i = (L(J \setminus \{\alpha_i\}); M[K]) \in D_{j-1,k}, \quad \text{for } 1 \leq i \leq j.$$

Then

$$\begin{aligned} \Phi \circ d'(x) &= (-1)^k \sum_{i=1}^j (-1)^{i-1} \Phi(x_i) \\ &= \sum_{K \in \mathcal{I}_r} \sum_{i=1}^j (-1)^{k+i-1} (L_{\alpha_1, \dots, \widehat{\alpha_i}, \dots, \alpha_j} \cap M_{\beta_1, \dots, \beta_k}) \\ &= (-1)^{j-1} \sum_{i=1}^j u(\Phi(x), \Phi(x_i))(\Phi(x)). \end{aligned}$$

Thus, whenever the compositions on $D_{j,k}$ make sense,

$$\Phi \circ d' = (-1)^{j-1} u \circ \Phi. \quad (18)$$

Similarly, letting $\tilde{x} = (L(J); \tilde{M}, M[K]) \in D_{j,k+1}$ we have

$$\begin{aligned} \Phi \circ d''(x) &= \sum'_{\tilde{M} \in \mathcal{M}_{k+1}, \tilde{M} \subset M_K} \Phi(\tilde{x}) \\ &= \sum'_{\substack{\beta_{k+1} \in [n] \setminus K \\ \tilde{M} \in \mathcal{M}_{k+1}, \tilde{M} \subset M_K}} (L_J \cap M_{K, \beta_{k+1}}) \\ &= \sum'_{\tilde{M} \in \mathcal{M}_{k+1}, \tilde{M} \subset M_K} v(\Phi(x), \Phi(\tilde{x})) \circ \Phi(x). \end{aligned}$$

Here $\sum'$ means that $L_J \cap M_{K, \beta_{k+1}}$ are all distinct. Thus, whenever the compositions make sense,

$$\Phi \circ d'' = v \circ \Phi. \quad (19)$$

Equations (18) and (19) show that the differentials of the complexes are compatible with Φ . In particular, using Proposition (3.5) we see that $d^2 = 0$. The proof of the proposition is now complete. $\square$

3.5. The Main Theorem

Recall that the q th rational motivic cohomology of an admissible pair of simplices $(L; M)$ is defined as

$$H_{\text{mot}}^q(L; M) = \bigoplus_{j=0}^n H^q(\mathbf{K}_{2j}^\bullet(L; M)),$$

where the complex $\mathbf{K}_{2j}^\bullet$ is constructed according to Definition 3.9. In this subsection we prove the following.

MAIN THEOREM 3.25. (I) *The complex $\mathbf{K}_{2j}^\bullet(L; M)$ is exact everywhere except at $\mathbf{K}_{2j}^0$. Moreover, $H^0(\mathbf{K}_0^\bullet(L; M)) \cong H^0(\mathbf{K}_{2n}^\bullet(L; M)) \cong \mathbb{Q}$ and $H^0(\mathbf{K}_{2j}^\bullet(L; M))$ has dimension at most $\binom{n}{j}^2$ for $0 < j < n$.*

(II) *Let $\mathfrak{S}_n$ denote the set of index subsets of $\{1, \dots, n\}$ in the increasing order. If $(L; M)$ is a generic pair then the set*

$$\{e(I, J) = (AL(0, I); M[J]) : |I| = j, |J| = n - j, I, J \in \mathfrak{S}_n\} \quad (20)$$

is a basis of $H^0(C_{2j}^\bullet(L; M))$. In particular, $H^0(\mathbf{K}_{2j}^\bullet(L; M))$ has dimension $\binom{n}{j}^2$. Here $(AL(K); \mathbb{P}^n) = AL(K)$ and $(\mathbb{P}^n; M_K) = (M_K)$ for any index set K .

REMARK 3.26. In the generic case, we will call L_0 and M_0 the special L -face and M -face respectively, when the basis is given as in the theorem. If $(L; M)$ is not a generic pair then it might be necessary to choose other faces than L_0 or M_0 as the special faces. Strictly speaking, the non-generic relations between L and M (for example, an M -vertex lies in some L -face) bring in extra relations among the otherwise linearly independent generators of the cohomology group. This is the reason why $H^0(\mathbf{K}_{2j}^\bullet(L; M))$ may have smaller dimension when $(L; M)$ is a non-generic pair. See the computations in the next three sections.

In what follows we will prove several lemmas before giving the proof of the Main Theorem at the end of this subsection. Thanks to Proposition 3.24 the theorem can be proved using either $C(L; M)$ or $\mathbf{K}(L; M)$ because we can always throw away torsions if necessary. We first handle the special cases $j = 0$ and n because they are in fact generic cases.

LEMMA 3.27. *One has $H^n(C_{2n}^\bullet(L; M)) = 0$. If $0 \leq k \leq n - 1$ then*

$$\ker d_{2n}^k = \langle AL(0, \alpha_1, \dots, \alpha_{n-k}) : 1 \leq \alpha_1 < \dots < \alpha_{n-k} \leq n \rangle.$$

Hence $H^q(C_{2n}^\bullet(L; M)) = 0$ unless $q = 0$ for which we have

$$H^0(C_{2n}^\bullet(L; M)) = \langle AL(0, \dots, n) \rangle.$$

Proof. It is clear that

$$C_{2n}^{n-1} = \langle (L_i) : 0 \leq i \leq n \rangle \longrightarrow C_{2n}^n = \langle (\mathbb{P}^n) \rangle$$

is surjective. Therefore $H^n(C_{2n}^\bullet(L; M)) = 0$.

Suppose $0 \leq k \leq n - 1$ and $d_{2n}^k(x) = 0$ for some $x \in C_{2n}^k = D_{n-k,0}$. By Proposition 3.24 we may write $x = x' + \sum_{1 \leq i_1 < \dots < i_k \leq n} a_{i_1, \dots, i_k} L(1, \dots, \hat{i}_1, \dots, \hat{i}_k, \dots, n)$ where L_0 appears in every term of x' . Since L_0 appears in every term of

$$x - \sum_{1 \leq i_1 < \dots < i_k \leq n} a_{i_1, \dots, i_k} AL(0, 1, \dots, \hat{i}_1, \dots, \hat{i}_k, \dots, n)$$

and by definition $AL(\dots) = d'(L(\dots))$, we may assume that $x = x'$, without loss of generality. Thus

$$x = \sum_{1 \leq i_0 < \dots < i_k \leq n} c_{i_0, \dots, i_k} L(0, \dots, \hat{i}_0, \dots, \hat{i}_k, \dots, n).$$

Now we can write $d_{2n}^k(x) = 0$ as

$$y + \sum_{1 \leq i_0 < \dots < i_k \leq n} c_{i_0, \dots, i_k} L(1, \dots, \widehat{i_0}, \dots, \widehat{i_k}, \dots, n) = 0$$

where L_0 appears in every term of y . Then we must have $y = 0$ by Proposition 3.24 and

$$\sum_{1 \leq i_0 < \dots < i_k \leq n} c_{i_0, \dots, i_k} L(1, \dots, \widehat{i_0}, \dots, \widehat{i_k}, \dots, n) = 0$$

which implies that $c_{i_0, \dots, i_k} = 0$ for all $i_0 < \dots < i_k$ because, again by Proposition 3.24, the $L(J)$ are linearly independent for all increasing index sets $J \in \llbracket n \rrbracket$. Thus $x = 0$. The lemma then follows from the fact that, for $1 \leq k \leq n-1$,

$$AL(0, \alpha_1, \dots, \alpha_{n-k}) = d_{2n}^{k-1}(L(0, \alpha_1, \dots, \alpha_{n-k})). \quad \square$$

The lemma implies that the complex $C_{2n}^\bullet$ is exact at C_{2n}^q whenever there is a non-trivial d' mapping to it. So the only exception is C_{2n}^0 and $AL(0, \dots, n)$ generates $H^0(C_{2n}^\bullet(L; M)) = \ker d_{2n}^0$. We will see that in the general situation we may apply the above argument to u on $\mathbf{K}(L; M)$.

LEMMA 3.28. *For all $0 \leq k \leq n-1$ we have*

$$\ker d_0^{k-n} = \left\langle \sum_{M^k \in \mathcal{M}_k} (M^k, M^{k-1}, \dots, M^1) : M^t \in \mathcal{M}_t, 1 \leq t \leq k-1 \right\rangle$$

where we set $\ker d_0^{-n} = 0$. Hence $H^q(C_0^\bullet(L; M)) = 0$ unless $q = 0$ for which we have $H^0(C_0^\bullet(L; M)) = \langle M[1, \dots, n] \rangle$.

Proof. This follows from Lemma 3.27 by taking the dual and using Proposition 3.11. $\square$

The lemma implies that the complex $C_0^\bullet$ is exact at C_0^q whenever it has a non-trivial d'' . So the only exception is at C_0^0 and $H^0(C_0^\bullet(L; M))$ is generated by $M[1, \dots, n]$. A similar result holds for v on $\mathbf{K}(L; M)$ in general.

Proof of the Main Theorem 3.25. By Lemma 3.17 all the D -groups in $C^\bullet(L; M)$ can be generated by generic pairs $(L^j; M^k)$. We now assume this.

Thanks to Lemmas 3.27 and 3.28 we may assume $0 < j < n$. We first look at the boundary cohomologies of $C_{2j}^\bullet(L; M)$. We deal with them as special cases because there is only one D -component at degree $j-n$ and j .

(i) When $k = j-n$ we have

$$C_{2j}^\bullet(L; M) : \dots \longrightarrow D_{0, n-j-1} \oplus D_{1, n-j} \xrightarrow{d_{2j}^{j-1}} D_{0, n-j} (= C_{2j}^j) \longrightarrow 0.$$

Clearly $D_{1, n-j}$ is not empty since $j \neq 0$. Therefore

$$d_{2j}^{j-1}(L_0; M^{n-j}, \dots, M^1) = (M^{n-j}, \dots, M^1)$$

implies that d_{2j}^{j-1} is surjective and $H^j(C_{2j}^\bullet(L; M)) = 0$.

(ii) When $k = j$ we have

$$C_{2j}^\bullet(L; M) : 0 \longrightarrow (C_{2j}^{j-n} =) D_{j,0} \xrightarrow{d_{2j}^{j-n}} D_{j-1,0} \oplus D_{j,1} \longrightarrow \dots$$

Note that $j < n$ so the component $D_{j,1} \neq 0$. Thus, for any $J = (\alpha_1, \dots, \alpha_j)$,

$$d_{2j}^{j-n}(L(J)) = \sum_{M^1 \in \mathcal{M}_1, M^1 \cap L_J \text{ distinct}} (L(J); M^1) \neq 0.$$

Hence $\ker d_{2j}^{j-n} = 0$ and d_{2j}^{j-n} is injective. Therefore $H^{j-n}(C_{2j}^\bullet(L; M)) = 0$.

In what follows we provide a proof, using $\mathbf{K}(L; M)$, which has a strong geometric flavor.

We only need to look at the terms $\mathcal{S}_{i,d}$ in the interior of each complex $\mathbf{K}_{2i}^\bullet(L; M)$ (for $1 \leq i \leq n-1$) extracted from Figure 1.

Suppose $(u+v)(\sum x_r) = 0$ where $x_r \in \mathcal{S}_{i,r}$ and (i, r) runs down the i th column of the 2-dimensional diagram in Figure 1, say $r_1 \geq r \geq r_2$. Then x_{r_1} is on the upper boundary of the parallelogram and we can apply Lemmas 3.28 and 3.27 and find that x_{r_1} is an image of an element sitting at the boundary of the $(i+1)$ th column. Thus we may assume $x_{r_1} = 0$. By induction on r we now may assume $x_r = 0$ for all $r_1 \geq r > r_2$. Let us prove that x_{r_2} is an image too. Suppose that

$$x_{r_2} = \sum_{(L_J; M_K) \in \text{OP}(L; M)} a_{(J; K)}(L_J \cap M_K)$$

with rational coefficients $a_{(J; K)}$. It is clear that either u or v is non-trivial because x_{r_2} is sitting on one of the two lower sides of the parallelogram defining $\mathbf{K}_{2i}^\bullet(L; M)$, but not on the vertices. Now let us consider the two cases separately.

(i) u is non-trivial. Set $J = (\alpha_1, \dots, \alpha_j)$. Then we have

$$u(x_{r_2}) = \sum_{(L_J; M_K) \in \text{OP}(L; M)} a_{(J; K)}(-1)^{k+j-i} \sum_{i=1}^j (L_{J \setminus \{\alpha_i\}} \cap M_K) = 0. \quad (21)$$

We observe that if $0 \notin J$ then $(L_{0,J} \cap M_K) \in \text{GP}(L; M)$. Indeed, if $(L_{0,J}; M_K)$ is degenerate then $L_{0,J} \cap M_K \subset L_0$ and we can replace the first entry in J by 0 which contradicts the optimality of $(L_J \cap M_K)$. Moreover, $(L_{0,J} \cap M_K) \in \text{OP}(L; M)$ because otherwise there is some $\tilde{J} \prec J$ with $(L_{0,J} \cap M_K) = (L_{0,\tilde{J}} \cap M_K)$ which implies that $(L_J \cap M_K) = (L_{\tilde{J}} \cap M_K)$ by definition, another contradiction. Now set

$$x'_{r_2} = x_{r_2} - (-1)^j \sum_{0 \notin J} a_{(J; K)} u((L_{0,J} \cap M_K))$$

where J ranges over those indices appearing in (21) (the sign $(-1)^j$ is to cancel all the terms $(L_J \cap M_K)$ satisfying $0 \notin J$ in x'_{r_2}). Then L_0 appears in every term of x'_{r_2} . Hence by forgetting the images we may assume that L_0 actually appears in every term of x_{r_2} . Rewriting J as $\{0, J\}$ with $J = (\alpha_2, \dots, \alpha_j)$ in (21) we get, by the assumption $u(x_{r_2}) = 0$,

$$\begin{aligned} 0 = u(x_{r_2}) &= \sum_{(L_{0,J}; M_K) \in \text{OP}(L; M)} a_{(0,J; K)}(-1)^{k+j-1} (L_J \cap M_K) \\ &+ \sum_{(L_{0,J}; M_K) \in \text{OP}(L; M)} a_{(0,J; K)}(-1)^{k+j-i} \sum_{i=2}^j (L_{0,J \setminus \{\alpha_i\}} \cap M_K). \end{aligned}$$

Notice that every term inside the second sum is optimal by Lemma 3.22 and therefore is linearly independent from the terms in the first sum which are also optimal. To prove this, notice that all the indices are in increasing order so that if some cancellation occurs between the first and second sum we must have $(L_J \cap M_K) \in L_0$ for some J . This contradicts the optimality of $(L_J \cap M_K)$. Thus

$$\sum_{(L_{0,J}; M_K) \in \text{OP}(L; M)} a_{(0,J;K)} (-1)^{k+j-1} (L_J \cap M_K) = 0.$$

This implies that all $a_{(0,J;K)} = 0$ because all the terms are optimal and therefore are linearly independent. Consequently $x_{r_2} = 0$.

(ii) v is non-trivial. This case is dual to (i) by Proposition 3.11.

The exactness claim in Part (I) of the theorem now follows immediately. Part (II) and the dimension estimate in (I) are straightforward and we leave the details to interested readers. $\square$

4. The coproduct on $A_\bullet \otimes \mathbb{Q}$

By Main Theorem 3.25 the cohomology $H_{\text{mot}}^q(L; M)$ is trivial for $a \neq 0$ and $G_j(L; M) = H^0(C_{2j}^\bullet(L; M))$ is free of rank $r_j \leq \binom{n}{j}^2$. Rewriting (3) for $q = 0$ using Corollary 3.24, we get

$$H_{\text{mot}}^0(L; M) = \bigoplus_{j=0}^n G_j(L; M)_{\mathbb{Q}}.$$

Let $\{e_{i,j} : 1 \leq i \leq r_j\}$ be a basis of $G_j(L; M)$. We expect that there exist well-defined maps

$$\varphi_{ij} : G_i(L; M)^* \otimes G_j(L; M) \longrightarrow A_{j-i}, \quad \text{for } 0 \leq i \leq j \leq n,$$

such that the coproduct map on A_n can be defined as

$$\begin{aligned} \nu_{k,n-k} : A_n &\longrightarrow A_{n-k} \otimes A_k, \\ [L; M] &\longmapsto \sum_{i=1}^{r_k} \varphi_{kn}[e_{i,k}^*] \otimes \varphi_{0k}[e_{i,k}], \end{aligned} \tag{22}$$

so that $A_{\bullet, \mathbb{Q}}$ becomes a graded commutative Hopf algebra. Here and in the rest of the paper,

$$\begin{aligned} \varphi_{0k}[e_{i,k}] &:= \varphi_{0k}[(M[1, \dots, n])^* \otimes e_{i,k}], \\ \varphi_{kn}[e_{i,k}^*] &:= \varphi_{kn}[e_{i,k}^* \otimes (AL(0, \dots, n))]. \end{aligned}$$

The maps φ_{ij} will also induce the actions

$$H_{\text{mot}}^q(L; M)_j \otimes A_{j-i, \mathbb{Q}}^* \longrightarrow H_{\text{mot}}^q(L; M)_i \tag{23}$$

which makes $H_{\text{mot}}^*(L; M)$ into a graded comodule over the graded Hopf algebra $A_{\bullet, \mathbb{Q}}$. Here if $q \neq 0$ and $i \neq j$ then it is the zero map. One can also define the Poincaré duality on the motivic cohomology (see [2, §2.18]).

4.1. The coproduct on the generic part

We carry out the above program in the generic case as follows.

DEFINITION 4.1. Let $(L; M)$ be a generic pair. For $0 \leq i \leq j \leq n$ we can define

$$\begin{aligned}\varphi_{ij} : G_i(L; M)^* \otimes G_j(L; M) &\longrightarrow A_{j-i}, \\ e(I, J)^* \otimes e(I', J) &\longmapsto \operatorname{sgn}(I)\operatorname{sgn}(J)[L_I \cap M_J | L_{0,I' \setminus I}; M_{0,J' \setminus J}],\end{aligned}$$

where $e(I, J)$ are defined in formula (20). Here for any $I \subset \{1, \dots, n\}$,

$$\operatorname{sgn}(I) = \operatorname{sgn}(I, \{1, \dots, n\} \setminus I).$$

LEMMA 4.2. *The maps φ_{ij} are well defined.*

Proof. The group $G_i(L; M)$ is defined modulo images of d_{2i}^{-1} which provide additivity and modulo the relations of the component groups of C_{2i}^0 which provides skew symmetry by Lemma 3.18. These two properties are also satisfied by A_{j-i} . $\square$

When $(i, j) = (0, k)$ or $(n - k, n)$ one gets

$$\begin{aligned}\varphi_{0k} : G_0(L; M)^* \otimes G_k(L; M) &\longrightarrow A_k, \\ (M[1, \dots, n])^* \otimes e(I, J) &\longmapsto \operatorname{sgn}(J)[M_J | L_{0,I}; M_{0,\bar{J}}]; \\ \varphi_{kn} : G_k(L; M)^* \otimes G_n(L; M) &\longrightarrow A_{n-k}, \\ e(I, J)^* \otimes (AL(0, \dots, n)) &\longmapsto \operatorname{sgn}(I)[L_I | L_{0,\bar{I}}; M_{0,J}],\end{aligned}$$

where $\bar{J} = \{1, \dots, n\} \setminus J$ and $\bar{I} = \{1, \dots, n\} \setminus I$ are in increasing order. This definition is essentially the same as the one given on page 708 of [3] (it was pointed out in [2] that j_k and j_{n-k} should be switched there). We would like to mention that everywhere the tensor product is over $\mathbb{Q}$ such that in the tensor product $A_m \otimes A_n$ the signs multiply. In particular, under the isomorphism $A_1 \cong F^\times$ the negative sign is mapped to the inverse map in $F^\times$.

DEFINITION 4.3. The coproduct maps are defined by

$$\begin{aligned}\nu_{n-k,k} : A_n^0 &\longrightarrow A_{n-k}^0 \otimes A_k^0, \\ [L; M] &\longmapsto \sum_{I,J} \varphi_{kn}[e(I, J)^*] \otimes \varphi_{0k}[e(I, J)].\end{aligned}$$

In the above definition we have fixed a basis for each $G_j(L; M)$. However, we have the following result.

PROPOSITION 4.4. *The coproduct map does not depend on the choice of the basis of $G_j(L; M)$ (for $0 \leq j \leq n$).*

Proof. This follows directly from Lemma 4.2. A combinatorial proof of this is provided as [13, Proposition 3.3]. $\square$

4.2. The coproduct on the non-generic part

Let $(L; M)$ be a non-generic pair of non-degenerate admissible simplices. By additivity on both L and M we may assume that non-generic conditions occur in only one L -flag and only one M -flag. We call such a pair *reduced*.

For any given $0 < k < n$ it is generally quite difficult to find a basis of $G_k(L; M)$ ($= H^0(C_{2k}^\bullet(L; M))$) because it depends crucially on the configuration of $(L; M)$. If a basis $\{e_{i,k} : 1 \leq i \leq \text{rk } G_k(L; M)\}$ has been found then φ_{0k} and φ_{kn} should be always definable in a way similar to the generic case. Then the coproduct would be given by (22). In fact, we believe that if $(L; M)$ is reduced then one can relabel L and M such that one of the following two cases should appear:

- (i) for each $I \subset \llbracket n \rrbracket$ and $|I| = k + 1$, there is a special M -face M_{j_I} such that each element in the basis of $G_k(L; M)$ has the form

$$e_{i,k} = e(I, J) = (AL(0, I); M[J]), \quad \text{where } I \subset \llbracket n \rrbracket \setminus \{0\}, J \subset \llbracket n \rrbracket \setminus \{j_I\};$$

- (ii) for each $J \subset \llbracket n \rrbracket$ and $|J| = n - k$, there is a special L -face L_{i_J} such that each element in the basis has the form

$$e_{i,k} = e(I, J) = (AL(i_J, I); M[J]), \quad \text{where } I \subset \llbracket n \rrbracket \setminus \{i_J\}, J \subset \llbracket n \rrbracket \setminus \{0\}.$$

Then in Case (i) setting $\bar{I} = \{1, \dots, n\} \setminus I$ and $\bar{J} = \{0, \dots, n\} \setminus (\{j_I\} \cup J)$ we define

$$\begin{aligned} \varphi_{0k}[e(I, J)] &= \text{sgn}(j_I, J, \bar{J})[M_J|L_{0,I}; M_{j_I}, \bar{J}], \\ \varphi_{kn}[e(I, J)^*] &= \text{sgn}(I)[L_I|L_{0,\bar{I}}; M_{j_I,J}]. \end{aligned}$$

In Case (ii) setting $\bar{I} = \{0, \dots, n\} \setminus (\{i_J\} \cup I)$ and $\bar{J} = \{1, \dots, n\} \setminus J$ we define

$$\begin{aligned} \varphi_{0k}[e(I, J)] &= \text{sgn}(J)[M_J|L_{i_J,I}; M_{0,\bar{J}}], \\ \varphi_{kn}[e(I, J)^*] &= \text{sgn}(i_J, I, \bar{I})[L_I|L_{i_J,\bar{I}}; M_{0,J}]. \end{aligned}$$

Note that in the generic case we have $i_J = j_I = 0$.

Applying the above idea in the next three sections we are going to deal with all possible configurations when $n = 2, 3$ and some non-generic configurations when $n = 4$, which settles these three cases explicitly.

5. Admissible pairs of triangles in $\mathbb{P}_F^2$

In this section, we use the theory developed in the previous sections to handle arbitrary configurations of admissible pairs of triangles in $\mathbb{P}_F^2$. This greatly simplifies the calculation given in [2, 4] for pairs of triangles. In particular, no auxiliary functions to define the coproduct map $\nu_{1,1}$ on A_2 will be needed because of the way we define φ_{01} and φ_{12} on the generators of $G_1(L; M)$.

5.1. Definition of coproduct on A_2

Let $[L; M] \in A_2$ be a non-degenerate admissible pair. If $(L; M)$ is not a generic pair then we further assume that the non-generic conditions occur in only one flag of L and only one flag of M . Hence we have the following three cases to consider:

- (I) $(L; M)$ is a generic pair;
- (II) $M_{01} \in L_1$;
- (III) $L_{01} \in M_1$.

Here, the condition in Case (II) is the only non-generic condition for the pair in that case. The same is true for Case (III).

In the following we only need to find the maps φ_{01} and φ_{12} by equation (22).

(I) $(L; M)$ is a generic pair. Then one has

$$G_1(L; M) = \langle (AL(0i); M_j) : i, j = 1, 2 \rangle.$$

The coproduct is defined in Definition 4.3.

(II) $M_{01} \in L_1$. Then

$$(L_1; M_i) \notin C_2^0 \text{ for } i = 0, 1 \text{ but } (L_1; M_0) + (L_1; M_1) \in C_2^0. \quad (24)$$

Hence

$$G_1 = \langle (AL(02); M_1); (AL(0i); M_2), (i = 1, 2) \rangle.$$

Thus $\text{rk } G_1 = 3$ and we can define

$$\begin{aligned} \varphi_{01} : (AL(02); M_1) &\mapsto [M_1|L_{0,2}; M_{0,2}], \\ (AL(01); M_2) &\mapsto -[M_2|L_{0,1}; M_{0,1}], \\ (AL(02); M_2) &\mapsto -[M_2|L_{0,2}; M_{0,1}], \\ \varphi_{12} : (AL(02); M_1) &\mapsto -[L_2|L_{0,1}; M_{0,1}], \\ (AL(01); M_2) &\mapsto [L_1|L_{0,2}; M_{0,2}], \\ (AL(02); M_2) &\mapsto -[L_2|L_{0,1}; M_{0,2}]. \end{aligned}$$

Here we abused the notation a little because what we really mean in the above is to define $\varphi_{01}((M[12])^*, -)$ and $\varphi_{01}(-, (AL(012)))$ where $(M[12])$ and $(AL(012))$ are the generators of $G_0(L; M)$ and $G_2(L; M)$ respectively. We will use the same notation in what follows.

By the general theory, $(AL(01); M_1)$ is another possible element in G_1 . But in our case $d_2^0 : (L_1; M_1) \mapsto -(M_0) - (M_1)$. Also notice that the image of this element under φ_{12} obtained in the generic situation, $[L_1|L_{0,2}; M_{0,1}]$, degenerates because $L_0 \cap M_0 = L_0 \cap M_1$. From the maps φ_{01} and φ_{12} we see that the coproduct $\nu_{1,1}([L; M])$ has only three terms. This is a general phenomenon, namely, the coproduct $\nu_{k,n-k}(L; M)$ always has $\text{rk } G_k(L; M)$ terms for admissible pairs $(L; M)$ in $\mathbb{P}^n$.

(III) $L_{01} \in M_1$. This can be treated by taking the dual (that is, exchange the letter L with M everywhere) in Case (II) due to Proposition 3.11.

5.2. Examples

As the first example we look at the pair of simplices $\Lambda_2(t) = (L; M)$ corresponding to the dilogarithms $Li_2(t)$ defined by Figure 4.

EXAMPLE 5.1 (Li_2). In Figure 4 one has $M_{01} \in L_1$, $L_{01} \in M_2$ and $L_{02} \in M_1$. Then using equation (24) and $(AL(02); M_1) = 0$ we can see that

$$G_1 = \langle (AL(02); M_2) \rangle.$$

Thus $\text{rk } G_1 = 1$ and we can define

$$\begin{aligned} \varphi_{01} : (AL(02); M_2) &\mapsto -[M_2|L_{0,2}; M_{0,1}] = -[\infty, 0; 1 - t, 1]; \\ \varphi_{12} : (AL(02); M_2) &\mapsto -[L_2|L_{0,1}; M_{0,2}] = -[\infty, 0; 1, t]. \end{aligned}$$

Notice that $[L_1|L_{0,2}; M_{0,1}]$, $[M_1|L_{0,2}; M_{0,2}]$ and $[M_2|dL_{0,1}; M_{0,1}]$ are degenerate.

 FIGURE 4. Dilogarithmic pair of simplices $\Lambda_2(t)$.

From equation (22) we get

$$\nu_{1,1}(\Lambda_2(t)) = -([\infty, 0; 1, t] \otimes [\infty, 0; 1, 1-t]) = -[t \otimes (1-t)]$$

under the isomorphism $A_1 \cong F^\times$ by the cross ratio. This agrees with [4, 2.14].

When $F = \mathbb{C}$ the configuration in this example corresponds to the classical dilogarithm function

$$Li_2(z) = \int_0^z \frac{dt}{1-t} \circ \frac{dt}{t}$$

which is an iterated integral from 0 to z in the sense of K.-T. Chen [5]. When $z = t$ is real this integral is equal to the 2-dimensional integral

$$\int_{0 < x < y < t} \frac{dx}{1-x} \frac{dy}{y} = - \int_{0 < 1-x < y < t} \frac{dx}{x} \frac{dy}{y}.$$

As the second example we look at the configuration corresponding to the double logarithms $Li_{1,1}$. See Figure 5.

EXAMPLE 5.2 ($Li_{1,1}$). In Figure 5 one has $L_{01} \in M_1$ and $L_{02} \in M_2$. It is easy to see that

$$G_1 = \langle (AL(01); M_2), (AL(02); M_1) \rangle.$$

By equation (22),

$$\begin{aligned} \nu_{1,1}([L; M]) = & [L_1|L_0, L_2; M_0, M_2] \otimes (-[M_2|L_0, L_1; M_0, M_1]) \\ & + (-[L_2|L_0, L_1; M_0, M_1]) \otimes [M_1|L_0, L_2; M_0, M_2]. \end{aligned}$$

To see the relation between this configuration and the double logarithm we take $F = \mathbb{C}$, $x = a_2/a_1$ and $y = 1/a_2$ where $a_1 \neq 0$ and $a_2 \neq 0, 1$. Then the double logarithm

$$Li_{1,1}(x, y) = \sum_{0 < m < n} \frac{x^m y^n}{mn}$$

can be analytically continued to $\mathbb{C}^2$ as a meromorphic function by the iterated integral in the sense of K.-T. Chen (see [16]). When x and y are real numbers this

FIGURE 5. Double-logarithmic pair of simplices.

integral becomes the 2-dimensional integral

$$\int_{0 < t_1 < t_2 < 1} \frac{dt_1}{t_1 - a_1} \frac{dt_2}{t_2 - a_2} = \int_{0 < t_1 + a_1 < t_2 + a_2 < 1} \frac{dt_1}{t_1} \frac{dt_2}{t_2}$$

corresponding to Figure 5. Notice that if $a_2 = 0$ then it becomes the classical dilogarithm treated in Example 5.1.

Let us first assume $a_1 \neq a_2$, that is, $x \neq 1$. By definition $a_1 \neq 0$ and $a_2 \neq 1$ and thus $[L; M]$ is admissible. So we have

$$\begin{aligned} \nu_{1,1}([L; M]) = & [\infty, 0; a_2 - a_1, -a_1] \otimes (-[\infty, 0; -a_2, 1 - a_2]) \\ & + (-[\infty, 0; a_1 - a_2, 1 - a_2]) \otimes [\infty, 0; 1 - a_1, -a_1] \end{aligned}$$

which corresponds to the formal equation (setting $Li_1(t) = -\log(1 - t)$)

$$\begin{aligned} & \int_0^{a_2} \frac{dt}{t - a_1} \otimes \int_0^1 \frac{dt}{t - a_2} + \int_{a_1}^1 \frac{dt}{t - a_2} \otimes \int_0^1 \frac{dt}{t - a_1} \\ & = Li_1(x) \otimes Li_1(y) + Li_1\left(\frac{1 - xy}{1 - x}\right) \otimes Li_1(xy). \end{aligned}$$

This is consistent with the double logarithm variation of mixed Hodge structures given by the matrix

$$\mathcal{M}_{1,1}(x, y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ Li_1(y) & 1 & 0 & 0 \\ Li_1(xy) & 0 & 1 & 0 \\ Li_{1,1}(x, y) & Li_1(x) & Li_1\left(\frac{1 - xy}{1 - x}\right) & 1 \end{pmatrix} \text{diag}[1, 2\pi i, 2\pi i, (2\pi i)^2].$$

This is essentially the same as defined in [8, § 2]. Notice that there is an error in that the term $2\pi i \log x$ in the matrix $\mathcal{M}_{1,1}(x, y)$ should be replaced by $2\pi i \log(1 - x)$. The explicit variations of mixed Hodge structures related to multiple logarithms $Li_{1,\dots,1}$ are given in [15].

It is interesting to see what happens when $a_1 = a_2 = t$, that is, $x = 1$. The configuration is the dual to Example 5.3. Setting

$$a = r(M_1|L_{1,0}; M_{0,2}) = r(0, \infty; 1 - t, -t)$$

one gets (recalling that $y = 1/t$), by the computation in Example 5.3,

$$\nu_{1,1}([L; M]) = -\nu_{1,1}([M; L]) = -[a \otimes (-a)] = -[\log(1 - y) \otimes (-\log(1 - y))].$$

On the other hand, when $x = 1$ we get the one-variable function

$$Li_{1,1}(1, y) = \sum_{0 < m < n} \frac{y^n}{mn}$$

which converges when $|y| < 1$. Then by an easy computation

$$dLi_{1,1}(1, y) = \sum_{0 < m < n} \frac{y^{n-1}}{m} dy = -\log(1 - y) d\log(1 - y).$$

We can compare this with the variation of mixed Hodge structures given by the matrix $\mathcal{M}_{1,1}(1, y)$ (see [15, p. 186]):

$$\mathcal{M}_{1,1}(1, y) = \begin{pmatrix} 1 & 0 & 0 \\ Li_1(y) & 1 & 0 \\ Li_{1,1}(1, y) & Li_1(y) & 1 \end{pmatrix} \text{diag}[1, 2\pi i, (2\pi i)^2].$$

As the last example in this subsection we treat the configuration denoted by $K(a)$ in [4, 2.13]. See Figure 6.

FIGURE 6. Configuration $K(a)$: $M_{01} \in L_2$, $M_{02} \in L_0$, and $M_{12} \in L_1$.

EXAMPLE 5.3. In Figure 6 one has $M_{01} \in L_2$, $M_{02} \in L_0$, and $M_{12} \in L_1$. It is not hard to see that

$$G_1 = \langle (L_2; M_2) + (L_1; M_0) + (L_0; M_1) - (\mathbb{P}^2) \rangle.$$

To calculate $\nu_{1,1}([L; M])$ we can cut M into three parts as follows. Let l_i be the L -vertex facing L_i and m_i the M -vertex facing M_i . Let $M_3 = \overline{l_1, m_0}$, $M_4 = \overline{l_2, m_2}$, $P = M_3 \cap M_4$, and $M_5 = \overline{P, m_1}$. By additivity we can express M as the sum of the oriented triangles

$$\Delta_1 = (M_2, M_5, M_3), \quad \Delta_2 = (M_0, M_4, M_5), \quad \Delta_3 = (M_1, M_3, M_4).$$

It is easy to see that $\nu_{1,1}(L; \Delta_3) = 0$ and

$$\nu_{1,1}(L; \Delta_1) = -[L_1|L_{2,0}; M_{2,5}] \otimes [M_5|L_{2,1}; M_{2,3}],$$

$$\nu_{1,1}(L; \Delta_2) = -[L_2|L_{1,0}; M_{0,5}] \otimes [M_5|L_{1,2}; M_{0,4}].$$

Set $A = L_1 \cap M_5$, $B = L_2 \cap M_5$ and $b = [M_5|L_{1,2}; M_{0,4}] = [A, B; m_1, P]$. Then clearly $[M_5|L_{2,1}; M_{2,3}] = [B, A; m_1, P] = b^{-1}$. By projecting from l_1 we have

$$[L_1|L_{2,0}; M_{2,5}] = [B, m_1; P, A] = \frac{-b}{1-b},$$

and by projecting from l_2 ,

$$[L_2|L_{1,0}; M_{0,5}] = [A, m_1; P, B] = \frac{1}{1-b}.$$

Thus

$$\nu_{1,1}([L; M]) = -\left(\frac{-b}{1-b} \otimes b^{-1}\right) - \left(\frac{1}{1-b} \otimes b\right) = (-b) \otimes b.$$

To express b intrinsically by quantities only from $(L; M)$ we follow [4, 2.13] to set $a = r(L_1|L_{0,2}; M_{1,0})$. By projective invariance, we can change L to the standard simplices and get the right-hand configuration of Figure 6. It is easy to see that $M_0 \cap L_1$ has coordinates $[a, 0, 1]$.

Let $P = [1, 1, 1]$. Then from the parallelogram (P, m_2, a, B) we get $|l_0, A| = a + 1$. By projecting to the x -axis one has

$$b = [A, B; m_1, P] = [a + 1, 0; \infty, 1] = -a^{-1}.$$

Therefore

$$\nu_{1,1}([L; M]) = a^{-1} \otimes (-a^{-1}) = (a \otimes a) - (a \otimes (-1)) = a \otimes (-a)$$

which agrees with [4, 2.13].

6. Admissible pairs of tetrahedra in $\mathbb{P}_F^3$

In this section, we calculate the motivic cohomology of all admissible pairs of simplices in A_3 from which we can easily define the coproduct

$$\nu = \bigoplus_{k=0}^3 \nu_{k,3-k} : A_3 \longrightarrow \bigoplus_{k=0}^3 A_k \otimes A_{3-k} \quad \text{where} \quad \nu_{k,3-k} : A_3 \longrightarrow A_k \otimes A_{3-k}$$

using equation (22). At the end of this section we will find the coproduct map on the multiple polylogarithmic pairs corresponding to $Li_{1,1,1}$, $Li_{2,1}$, $Li_{1,2}$ and Li_3 as corollaries.

6.1. Definition of coproduct on A_3

Let $[L; M] \in A_3$ be a non-degenerate admissible pair. As for A_2 , by additivity on both L and M we have the following twelve cases to consider:

(I) $(L; M)$ is a generic pair,	(II) $M_{012} \in L_1$,
(III) $M_{012} \in L_{12}$,	(IV) $M_{01} \subset L_1$ and $M_{012} \in L_{12}$,
(V) $M_{012} \in L_{12} \subset M_1$,	(VI) $L_{012} \in M_1$,
(VII) $L_{012} \in M_{12}$,	(VIII) $L_{01} \subset M_1$ and $L_{012} \in M_{12}$,
(IX) $L_{012} \in M_{12} \subset L_1$,	(X) $M_{01} \cap L_{01} \neq \emptyset$,
(XI) $M_{01} \in L_1$,	(XII) $L_{01} \in M_1$,

where the condition(s) in each of the above cases are the only non-generic conditions for the pair in that case.

In view of the duality of L and M provided by Proposition 3.11, we can reduce (VI) to (II), (VII) to (III), (VIII) to (IV), and (IX) to (V). Further, Case (X) can be reduced to Case (IV): Let $E = M_{01} \cap L_{01}$. By additivity M is a sum (or difference) of M' and M'' where we get M' and M'' by replacing M_{012} and M_{123} , respectively, with E . Then $(L; M)$ is a sum (or difference) of two pairs in Case (IV). Similarly, (XI) can be reduced to Case (V) and (XII) can be reduced to (VI).

We deal with Case (I) to (V) separately. We only need to show that either Case (i) or Case (ii) in § 4.2 appears, so that we may define the maps φ_{01} , φ_{13} , φ_{02} and φ_{23} by the procedure presented in § 4.2. Then the coproduct is defined by equation (22).

(I) $(L; M)$ is a generic pair. Then one has

$$G_1(L; M) = \langle (AL(0k); M[ij]) : 1 \leq i < j \leq 3, k = 1, 2, 3 \rangle,$$

and

$$G_2(L; M) = \langle (AL(0ij); M_k) : 1 \leq i < j \leq 3, k = 1, 2, 3 \rangle.$$

The coproduct is defined in Definition 4.3.

(II) $M_{012} \in L_1$. Then

$$\begin{aligned} (L_1; M[ij]) &\notin C_2^0, \text{ for } 0 \leq i, j \leq 2; & (L_1; M[01]) + (L_1; M[02]) &\in C_2^0, \\ (L_1; M[10]) + (L_1; M[12]) &\in C_2^0, & (L_1; M[20]) + (L_1; M[21]) &\in C_2^0. \end{aligned}$$

One has

$$M_{012} \in L_1 \implies (AL(01); M[ij]) \notin \ker d_2^0, \text{ for } 0 \leq i, j \leq 2. \quad (25)$$

Hence $\text{rk } G_1 = 8$ and

$$G_1 = \langle (AL(0k); M[i3]), i = 1, 2, k = 1, 2, 3; (AL(0k); M[12]), k = 2, 3 \rangle.$$

Therefore one can define

$$\begin{aligned} \varphi_{01} : (AL(0k); M[ij]) &\longmapsto (-1)^{c(ij)-1} [M_{ij}|L_{0,k}; M_{0,c(ij)}], \\ \varphi_{13} : (AL(0k); M[ij]) &\longmapsto (-1)^{k-1} [L_k|L_{0,c(k)}; M_{0,i,j}]. \end{aligned}$$

Here for any index subset S of $\{1, 2, 3\}$ in increasing order, $c(S)$ is the complementary subset in increasing order.

By the general theory $(AL(01); M[12])$ is another possible element in G_1 . But in our case $L_1 \cap M_{01} = L_1 \cap M_{02} = L_1 \cap M_{12}$ implies that $[L_1|L_{0,2,3}; M_{0,1,2}]$ is degenerate. Therefore there are only eight terms in $\nu_{1,2}([L; M])$.

Since the condition clearly has no effect on $G_2(L; M)$ by definition, we see that $\text{rk } G_2 = 9$ and $\nu_{2,1}([L; M])$ is defined exactly the same as in the generic case.

(III) $M_{012} \in L_{12}$. Then $M_{012} \in L_1$ and $M_{012} \in L_2$, so one can use Case (I) to deduce that

$$\begin{aligned} G_1(L; M) &= \langle (L_0; M[i3]) - (L_k; M[i3]), i = 1, 2, k = 1, 2, 3; \\ &\quad (L_0; M[12]) - (L_3; M[12]) \rangle. \end{aligned}$$

Thus $\text{rk } G_1 = 7$, so there are exactly seven terms in $\nu_{1,2}([L; M])$. Note that $[L_1|L_{0,2,3}; M_{0,1,2}]$ and $[L_2|L_{0,1,3}; M_{0,1,2}]$ are degenerate.

To find the basis of $G_2(L; M)$ we notice that, for $i = 1, 2$,

$$(L_{12}, L_i; M_k) \notin C_4^0, \text{ for } k = 0, 1, 2; \quad \text{but } \sum_{k=0}^2 (L_{12}, L_i; M_k) = 0.$$

We see that

$$M_{012} \in L_{12} \implies (AL(012); M_k) \notin \ker d_4^0 \text{ for } k = 0, 1, 2. \quad (26)$$

Therefore

$$G_2(L; M) = \langle (AL(0i3); M_k), i = 1, 2, k = 1, 2, 3; (AL(012); M_3) \rangle.$$

Therefore $\text{rk } G_2 = 7$. Thus there are exactly seven terms in $\nu_{2,1}([L; M])$ also. Note that for $k = 1, 2$, both $[L_{12}|L_{0,3}; M_{0,k}]$ are degenerate.

(IV) $M_{01} \subset L_1$ and $M_{012} \in L_{12}$. Then

$$M_{01} \subset L_1 \implies \begin{cases} (L_1; M[i2]) \notin C_2^0, i = 0, 1; \sum_{i=0}^1 (L_1; M[i2]) \in C_2^0; \\ (L_1; M[i3]) \notin C_2^0, i = 0, 1; \sum_{i=0}^1 (L_1; M[i3]) \in C_2^0; \end{cases} \quad (27)$$

and from Case (III) one can deduce that

$$G_1(L; M) = \langle (AL(0k); M[ij]), 1 \leq i, j, k \leq 3, \\ (i, j, k) \neq (1, 2, 1), (1, 2, 2), (1, 3, 1) \rangle.$$

Hence $\text{rk } G_1 = 6$ and there are exactly six terms in $\nu_{1,2}([L; M])$. Note that the following are all degenerate: $[L_1|L_{0,2,3}; M_{0,1,2}]$, $[L_1|L_{0,2,3}; M_{0,1,3}]$ and $[L_2|L_{0,2,3}; M_{0,1,2}]$.

To find the basis of $G_2(L; M)$ one can use

$$M_{01} \subset L_1 \implies \begin{cases} (AL(01i); M_1) \notin \ker d_4^0 \text{ for } i = 2, 3; \\ (AL(012); M_1) + (AL(013); M_1) \in \ker d_4^0, \end{cases} \quad (28)$$

and equation (26) to get

$$G_2(L; M) = \langle (AL(012); M_3); (AL(013); M_k), k = 2, 3; \\ (AL(023); M_k), k = 1, 2, 3 \rangle.$$

Therefore $\text{rk } G_2 = 6$. Thus there are exactly six terms in $\nu_{2,1}([L; M])$ also. Note that for $k = 2, 3$, $[L_{1k}|L_{0,c(1,k)}; M_{0,1}]$ are degenerate, and so is $[L_{12}|L_{0,3}; M_{0,2}]$.

(V) $M_{012} \in L_{12} \subset M_1$. Then one can use Case (III) to get

$$G_1(L; M) = \langle (AL(03); M[12]); (AL(1k); M[13]), k = 0, 3; \\ (AL(0k); M[23]), k = 1, 2, 3 \rangle.$$

Thus $\text{rk } G_1 = 6$, so there are exactly six terms in $\nu_{1,2}([L; M])$. Clearly there is no way one can use the formula in the generic case. We have to choose L_1 as the special plane for M_{13} which belongs to Case (ii) in § 4.2. Hence we get

$$\begin{aligned} \varphi_{01} : (AL(10); M[13]) &\mapsto -[M_{13}|L_{1,0}; M_{0,2}], \\ (AL(13); M[13]) &\mapsto -[M_{13}|L_{1,3}; M_{0,2}]; \\ \varphi_{13} : (AL(10); M[13]) &\mapsto -[L_0|L_{1,2,3}; M_{0,1,3}], \\ (AL(13); M[13]) &\mapsto -[L_3|L_{1,0,2}; M_{0,1,3}]. \end{aligned}$$

The missing terms $[L_k|\dots; M_{0,1,2}]$ for $k = 1, 2$ and $[M_{13}|L_{1,2}; M_{0,2}]$ are degenerate.

We now have

$$L_{12} \subset M_1 \implies (AL(12i), M_1) = 0 \text{ for } i = 2, 3. \quad (29)$$

From equation (26) we know that $(AL(012), M_2) \notin \ker d_4^0$. Thus

$$G_2(L; M) = \langle (AL(103); M_1); (AL(0i3); M_2), i = 1, 2; \\ (AL(0ij); M_3), 1 \leq i, j \leq 3 \rangle.$$

Therefore $\text{rk } G_2 = 6$ also. Hence there are exactly six terms in $\nu_{2,1}([L; M])$ also. We define:

$$\begin{aligned} \varphi_{02} : (AL(103); M_1) &\longmapsto [M_1|L_{1,0,3}; M_{0,2,3}], \\ (AL(0i3); M_2) &\longmapsto -[M_2|L_{0,i,3}; M_{0,1,3}], \quad i = 1, 2, \\ (AL(0ij); M_3) &\longmapsto [M_3|L_{0,i,j}; M_{0,1,2}], \\ \varphi_{23} : (AL(103); M_1) &\longmapsto [L_{03}|L_{1,2}; M_{0,1}], \\ (AL(0i3); M_2) &\longmapsto (-1)^i [L_{i,3}|L_{0,c(i3)}; M_{0,2}], \quad i = 1, 2, \\ (AL(0ij); M_3) &\longmapsto (-1)^{c(ij)-1} [L_{ij}|L_{0,c(ij)}; M_{0,3}]. \end{aligned}$$

Note that for M_1 we choose L_1 as the special L -face and for M_2 and M_3 we choose L_0 as usual. Hence we are in Case (ii) of § 4.2. After this choice of special L -faces one sees that $[M_1|L_{1,0,2}; M_{0,2,3}]$, $[M_1|L_{1,2,3}; M_{0,2,3}]$ and $[L_{12}|L_{0,3}; M_{0,2}]$ are degenerate.

6.2. Examples

As applications of our general theory presented in the previous subsection let us find the images of the coproduct map on the multiple polylogarithmic pairs of simplices of weight 3.

EXAMPLE 6.1 ($Li_{1,1,1}$). The triple logarithm is defined as

$$Li_{1,1,1}(x, y, z) = \sum_{0 < l < m < n} \frac{x^l y^m z^n}{lmn} \quad \text{for } |x|, |y|, |z| < 1.$$

It can be extended to a meromorphic function on $\mathbb{C}^3$ by iterated integrals (cf. [16]) by setting $(x, y, z) = (a_2/a_1, a_3/a_2, 1/a_3) \neq (0, 0, 0)$:

$$Li_{1,1,1}(x, y, z) = - \int_0^1 \frac{dt}{t - a_1} \circ \frac{dt}{t - a_2} \circ \frac{dt}{t - a_3}.$$

When x, y and z are real this becomes a 3-dimensional integral

$$- \int_{0 < t_1 < t_2 < t_3 < 1} \frac{dt_1}{t_1 - a_1} \frac{dt_2}{t_2 - a_2} \frac{dt_3}{t_3 - a_3}. \quad (30)$$

Here we have assumed that $a_i \neq 0$. By substitution $t_i \rightarrow t_i + a_i$ and taking $L_1 = \{t_2 = 0\}$, $L_2 = \{t_1 = 0\}$, $L_3 = \{t_3 = 0\}$ and $L_0 = \{t_0 = 0\}$ as the infinite plane we see that the above iterated integral corresponds to an admissible pair of tetrahedra $(L; M(\mathbf{a}))$ where $\mathbf{a} = [1, a_1, a_2, a_3]$ in $\mathbb{P}_{\mathbb{C}}^3$ (see Figure 7). In affine coordinates the region bounded by the tetrahedron $M(\mathbf{a})$ is a translation of the region $0 < t_1 < t_2 < t_3 < 1$ by translating $(0, 0, 0)$ to the coordinate $(-a_1, -a_2, -a_3)$.

FIGURE 7. Triple logarithmic pairs of simplices.

For arbitrary field F we can put L as the standard simplex with L_1 and L_2 exchanged (because of the negative sign in equation (30)) and define $M = (M_0, \dots, M_3)$ in $\mathbb{P}_F^3$ as follows: write m_i as the vertex facing M_i and set

$$\begin{aligned} m_0 &= [1, a_1, a_2, a_3 + 1], & m_1 &= [1, a_1 + 1, a_2 + 1, a_3 + 1], \\ m_2 &= [1, a_1, a_2 + 1, a_3 + 1], & m_3 &= [1, a_1, a_2, a_3]. \end{aligned}$$

We see that the following conditions are satisfied:

$$L_{02} \subset M_1, \quad L_{03} \subset M_3, \quad L_{013} \in M_{03}, \quad L_{012} \in M_{12}, \quad L_{023} \in M_{13}.$$

Because $(AL(01); M[30]) = 0$ (note that $L_0 \cap M_{30} = L_1 \cap M_{30}$) and

$$(L_i; M[32]) + (L_i; M[31]) + (L_i; M[30]) = d_2^{-1}((L_i; M_3)) \quad \text{for } i = 0, 1,$$

we see that

$$\begin{aligned} G_1(L; M(\mathbf{a})) &= \langle a = (AL(02); M[30]); b = (AL(03); M[21]); \\ &\quad c = (AL(01); M[31]) = -(AL(01); M[32]) \bmod \text{im}(d_2^{-1}) \rangle. \end{aligned}$$

Hence $\text{rk } G_1(L; M(\mathbf{a})) = 3$. Now we can define

$$\begin{aligned} \varphi_{01} : a &\mapsto [M_{30}|L_{0,2}; M_{2,1}] = [\infty, 0; 1 - a_1, -a_1], \\ b &\mapsto [M_{21}|L_{0,3}; M_{0,3}] = -[\infty, 0; -a_3, 1 - a_3], \\ c &\mapsto [M_{31}|L_{0,1}; M_{0,2}] = [\infty, 0; 1 - a_2, -a_2]; \\ \varphi_{13} : a &\mapsto -[L_2|L_{0,1,3}; M_{2,3,0}], \\ b &\mapsto [L_3|L_{0,1,2}; M_{0,2,1}], \\ c &\mapsto \mu([L_1 \cap M_3|L_{0,2}; M_{2,1}] \otimes [L_1 \cap M_1|L_{0,3}; M_{3,0}]) \\ &= [\infty, 0; a_2 - a_1, -a_1] \cdot [\infty, 0; 1 - a_3, a_2 - a_3]. \end{aligned}$$

We get $\varphi_{13}(c)$ by cutting $M(\mathbf{a})$ into two parts along M_{02} . Note that $[M_{12}|L_{0,1}; M_{0,3}]$

and $[M_{i3}|L_{0,3}; \dots]$ for $i = 1, 2$ are degenerate. For L_2 we choose M_2 as the special M -face since both $[M_{13}|L_{0,2}; M_{2,0}]$ and $[M_{01}|L_{0,2}; M_{2,3}]$ are degenerate.

By using equations (29) and

$$L_{012} \in M_1 \implies (AL(012), M_1) = 0 \quad (31)$$

it is not too hard to find

$$\begin{aligned} G_2(L; M(\mathbf{a})) &= \langle (AL(023); M_2) = -(AL(023); M_0); \\ &\quad (AL(013); M_2) = -(AL(013); M_1); \\ &\quad (AL(012); M_3) = -(AL(012); M_0) \rangle. \end{aligned}$$

So $\text{rk } G_2(L; M(\mathbf{a})) = 3$ and we can define

$$\begin{aligned} \varphi_{02} : (AL(023); M_2) &\longmapsto -[M_2|L_{0,2,3}; M_{0,1,3}], \\ (AL(013); M_2) &\longmapsto [M_2|L_{0,1,3}; M_{1,0,3}], \\ (AL(012); M_3) &\longmapsto [M_3|L_{0,1,2}; M_{0,1,2}], \\ \varphi_{23} : (AL(023); M_2) &\longmapsto [L_{23}|L_{0,1}; M_{0,2}], \\ (AL(013); M_2) &\longmapsto -[L_{13}|L_{0,2}; M_{1,2}], \\ (AL(012); M_3) &\longmapsto [L_{12}|L_{0,3}; M_{0,3}]. \end{aligned}$$

Note that we use M_1 as the special M -face when we define $\varphi_{23}(AL(013); M_k)$ because when $k = 0, 3$ they are degenerate due to the inclusion $L_{013} \in M_{03}$. Similarly one can show that φ_{02} sends all other possible candidates of the basis of G_2 to zero: $[M_k|L_{0,1,2}; \dots]$ for $k = 1, 2$ and $[M_k|L_{0,2,3}; \dots]$ for $k = 1, 3$ are degenerate.

The coproducts produced by the above maps $\varphi_{i,j}$ using formula (22) agree with the variations of the mixed Hodge structures related to $Li_{1,1,1}$ obtained by using a different technique (see [15]).

EXAMPLE 6.2 ($Li_{1,2}$). If we let $a_3 = 0$ in the preceding example we get $Li_{1,2}$. There is only one more condition $M_{012} \in L_3$. This does not affect G_2 so $\text{rk } G_2 = 3$. But

$$M_{012} \in L_3 \implies (AL(03); M[12]) \notin C_2^0,$$

so that

$$G_1(L; M) = \langle a = (AL(02); M[30]); c = (AL(01); M[31]) \rangle.$$

EXAMPLE 6.3 ($Li_{2,1}$). If we let $a_2 = 0$ in Example 6.1 we will get $Li_{2,1}$. The one more restriction is $M_{12} \subset L_1$ which does not affect G_1 as given in Example 6.2. But from equation (28), $(AL(013); M_1) \notin \ker d_4^0$, one has

$$\begin{aligned} G_2(L; M(\mathbf{a})) &= \langle (AL(023); M_2) = -(AL(023); M_0); \\ &\quad (AL(012); M_3) = -(AL(012); M_0) \rangle. \end{aligned}$$

So $\text{rk } G_1 = \text{rk } G_2 = 2$.

EXAMPLE 6.4 (Li_3). If we let $a_2 = a_3 = 0$ in Example 6.1 we get Li_3 , the classical trilogarithm. The additional restrictions are $M_{12} \subset L_1$, $M_{012} \in L_{13}$ and $L_{13} \subset M_0$ (see Figure 8).

FIGURE 8. Trilogarithmic pairs of simplices.

We see that b of Example 6.1 is not in $\ker d_2^0$ since, by equation (25),

$$M_{012} \in L_3 \implies (L_3; M[ij]) \notin C_2^0 \text{ for } 0 \leq i, j \leq 2.$$

Also c of Example 6.1 is not in $\ker d_2^0$ because, by equation (27),

$$M_{12} \subset L_1 \implies (L_1; M[i3]) \notin C_2^0 \text{ for } i = 1, 2.$$

Hence one has

$$G_1(L; M(\mathbf{a})) = \langle a = (AL(02); M[30]) \rangle.$$

Now from equation (29), $L_{13} \subset M_0$ implies that $(AL(132); M_0) = 0$, and from (31), $L_{13} \subset M_0$ implies that

$$(AL(012); M_0) = (AL(032); M_0) = -(AL(023); M_0).$$

Therefore $G_2(L; M(\mathbf{a}))$ is generated by

$$\langle (AL(012); M_3) = -(AL(012); M_0) = (AL(023); M_0) = -(AL(023); M_2) \rangle.$$

The upshot is that $\text{rk } G_1 = \text{rk } G_2 = 1$ and

$$\begin{aligned} \nu_{2,1}([L; M]) &= \varphi_{13}(a) \otimes \varphi_{01}(a) \\ &= (-[L_2|L_{0,1,3}; M_{2,3,0}]) \otimes [M_{30}|L_{0,2}; M_{2,1}] \\ &= \frac{1}{2}\mu([\infty, 0; 1, a_1] \otimes [\infty, 0; 1, a_1]) \otimes [\infty, 0; 1 - a_1, -a_1], \\ \nu_{1,2}([L; M]) &= \varphi_{23}((AL(012); M_3)) \otimes \varphi_{02}((AL(012); M_3)) \\ &= [\infty, 0; 1, a_1] \otimes [M_3|L_{0,1,2}; M_{0,1,2}]. \end{aligned}$$

Here we have used bary-center subdivision of M to find

$$\varphi_{23}((AL(012); M_3)) = [\infty, 0; 1, a_1].$$

Let $\Lambda_3(x) = (L; M)$ be as given in Figure 8. It represents the element in $A_3(F)$ which corresponds to $Li_3(x)$ when $F = \mathbb{C}$. Then we see that $[M_3|L_{0,1,2}; M_{0,1,2}] = -\Lambda_2(x)$

which is the dilogarithmic pair (see Figure 4) and

$$\begin{aligned}\nu_{2,1}(\Lambda_3(x)) &= -[\tfrac{1}{2}x^2 \otimes (1-x)], \\ \nu_{1,2}(\Lambda_3(x)) &= -[x \otimes \Lambda_2(x)],\end{aligned}$$

under the identification $A_1 \cong F^\times$ by the cross ratio.

The coproducts are consistent with the well-known variation matrix of the mixed Hodge structures related to the trilogarithm (see [11])

$$\mathcal{M}_3(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\log(1-x) & 1 & 0 & 0 \\ Li_2(x) & \log x & 1 & 0 \\ Li_3(x) & \frac{1}{2}(\log x)^2 & \log x & 1 \end{pmatrix} \text{diag}[1, 2\pi i, (2\pi i)^2, (2\pi i)^3].$$

7. Admissible pairs of simplices in $\mathbb{P}_F^4$

Using a procedure similar to the one adopted in the previous section one can define the coproduct on A_4 without too much difficulty. Nevertheless, to make the exposition shorter we content ourselves with only two examples in this case. To define the coproduct on A_n (for $n \geq 5$) in general is conceivably much more difficult.

EXAMPLE 7.1 (Part of the tetralogarithm Li_4). When x is real, the tetralogarithm $Li_4(x)$ can be defined by the 4-dimensional integral

$$Li_4(x) = - \int_{0 < 1-t_1 < t_2 < t_3 < t_4 < x} \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \wedge \frac{dt_3}{t_3} \wedge \frac{dt_4}{t_4}.$$

Take L to be the standard simplex under the odd permutation:

$$L_0 \longrightarrow L_3 \longrightarrow L_1 \longrightarrow L_4 \longrightarrow L_0.$$

Then the region of the above integration is defined by the pair of simplices $\Lambda_4(x) = (L; M)$ where M is given by:

$$\begin{aligned}M_0 : \{t_4 = t_3\}, \quad M_1 : \{t_3 - t_4 = t_2\}, \quad M_2 : \{t_2 = t_1\}, \\ M_3 : \{t_1 = t_0\}, \quad M_4 : \{t_0 = xt_3\}.\end{aligned}$$

Recall that for any $(L; M) \in A_4$ we may apply additivity on both M and L using a method similar to bary-center subdivision such that $(L; M)$ is decomposed into reduced pairs. The inclusion conditions of the subdivisions of $\Lambda_4(x)$ are mostly similar to the ones we encountered in A_3 , except the following:

$$M_{0123} \in L_{012} \subset L_{01} \subset M_1.$$

In what follows we show how to define the coproduct on this piece of subdivision of $(L; M)$.

One has

$$M_{0123} \in L_{012} \implies (L_0; M[123]) = (L_1; M[123]) = (L_2; M[123])$$

and

$$L_{01} \subset M_1 \implies (AL(01); M[1kl]) = 0, \text{ for } 2 \leq k < l \leq 4.$$

Hence

$$G_1 = \langle (AL(01); M[234]); (AL(02); M[jkl]), (j, k, l) \neq (1, 2, 3); \\ (AL(0i); M[jkl]), i = 3, 4, 1 \leq j < k < l \leq 4 \rangle.$$

Thus $\text{rk } G_1 = 12$. Note that $[L_2 | \dots; M_{0,1,2,3}]$ and $[M_{1kl} | L_{0,1}; \dots]$ (for $2 \leq k < l \leq 4$) are degenerate.

We can see that

$$M_{0123} \in L_{012} \implies (AL(012); M[kl]) = 0, \text{ for all } 0 \leq k, l \leq 3, \quad (32)$$

and

$$L_{01} \subset M_1 \implies (AL(01i); M[1k]) = 0, \text{ for } 2 \leq j, k \leq 4.$$

These yield

$$G_2 = \langle (AL(012); M[k4]), k = 2, 3; \\ (AL(01i); M[kl]), i = 3, 4, 2 \leq k < l \leq 4; \\ (AL(0ij); M[kl]), 2 \leq i < j \leq 4, 1 \leq k < l \leq 4 \rangle.$$

Thus $\text{rk } G_2 = 26$. Note that $[M_{23} | L_{0,1,2}; \dots]$ is degenerate and for any $2 \leq i, k \leq 4$, $[M_{1k} | L_{0,1,i}; \dots]$ are degenerate too.

To treat $G_3(L; M)$ one has

$$M_{0123} \in L_{012} \implies (AL(012k); M_l) \notin \ker d_6^0, \text{ for all } 0 \leq l \leq 3.$$

It is clear that

$$L_{01} \subset M_1 \implies (AL(01jk), M_1) = 0.$$

Thus

$$G_3 = \langle (AL(0234); M_1); (AL(0ijk); M_4), 1 \leq i < j < k \leq 4; \\ (AL(3ij4); M_l), 0 \leq i < j \leq 2, l = 2, 3 \rangle.$$

Thus $\text{rk } G_3 = 11$. Note that $[L_{012} | \dots; M_{0,l}]$ (for $l = 2, 3$) and $[M_1 | L_{01jk}; \dots]$ are degenerate. Here for fixed M_2 and M_3 we used L_3 as the special L -face.

EXAMPLE 7.2. Suppose $(L; M)$ is an admissible pair of simplices in $\mathbb{P}_F^4$ with only the following non-generic conditions:

$$M_{0123} \in L_{012} \subset M_{01} \subset L_1. \quad (33)$$

Together with its dual condition (L and M exchanged), such a configuration is essentially the only obstacle to expressing Aomoto tetralogarithms as linear combinations of the classical tetralogarithms together with products of polylogarithms of lower weights, because no such condition as (33) appears in the configurations corresponding to polylogarithms or their products. We sketch the proof here. We know that A_2 is generated by dilogarithmic pairs $\Lambda_2(x)$ and squares [2] and in [14, § 4.7] we proved that A_3 is generated by trilogarithmic pairs $\Lambda_3(y)$ and prisms. So we only need to look into the following five cases:

- (i) $\Lambda_4(x)$ (Example 7.1),
- (ii) $\mu(x \otimes \Lambda_3(y))$,
- (iii) $\mu(\Lambda_2(x) \otimes \Lambda_2(y))$,
- (iv) $\mu(x \otimes y \otimes \Lambda_2(z))$ and
- (v) $\mu(x \otimes y \otimes z \otimes w)$

for x, y, z and w in $F^\times$. Then we can easily check that the condition (33) cannot appear in any of the above five cases.

Now we determine the group $G_1(L; M)$. From

$$M_{01} \subset L_1 \implies (L_1; M[0kl]) \notin C_2^0, (L_1; M[1kl]) \notin \ker d_2^0$$

(but $(L_1; M[0kl]) + (L_1; M[1kl]) \in C_2^0$) and

$$M_{0123} \in L_{012} \implies (L_i; M[123]) \notin \ker d_2^0, \quad \text{for } i = 0, 1, 2,$$

one has

$$G_1 = \langle (AL(34); M[123]); (AL(0i); M[234]), 1 \leq i \leq 4; \\ (AL(0i); M[1k4]), i = 2, 3, 4, k = 2, 3 \rangle.$$

Thus $\text{rk } G_1 = 11$. Note that $[L_1 | \dots; M_{0,1,k,4}]$ (for $k = 2, 3$), and $[L_i | \dots; M_{0,1,2,3}]$ (for $i = 0, 1, 2$) are degenerate. Here for fixed $M_{1,2,3}$ we used L_3 as the special L -face.

To determine $G_2(L; M)$ and $G_3(L; M)$ one can use relation (32) and

$$M_{01} \subset L_1 \implies (AL(1ij); M[0k]) \notin \ker d_4^0, (AL(1ij); M[1k]) \notin d_4^0,$$

but

$$(AL(1ij); M[0k]) + (AL(1ij); M[1k]) \in \ker d_4^0.$$

Then

$$G_2 = \langle (AL(012); M[k4]), k = 2, 3; \\ (AL(01i); M[kl]), i = 3, 4, 2 \leq k < l \leq 4; \\ (AL(0ij); M[kl]), 2 \leq i < j \leq 4, 1 \leq k < l \leq 4 \rangle.$$

Thus $\text{rk } G_2 = 26$. Note that $[M_{23} | L_{0,1,2}; \dots]$ is degenerate and for any $2 \leq i, k \leq 4$, $[L_{1i} | \dots; M_{0,1,k}]$ are degenerate too.

We now turn to $G_3(L; M)$. Because

$$M_{0123} \in L_{012} \implies (AL(012i); M_l) \notin \ker d_6^0, \quad \text{for } 0 \leq l \leq 3,$$

and

$$M_{01} \subset L_1 \implies (AL(1ijk); M_l) \notin \ker d_6^0, \quad \text{for } l = 0, 1,$$

one has

$$G_3 = \langle (AL(0234); M_1); (AL(0ijk); M_4), 1 \leq i < j < k \leq 4; \\ (AL(3ij4); M_l), 0 \leq i < j \leq 2, l = 2, 3 \rangle.$$

Thus $\text{rk } G_3 = 11$. Note that $[L_{1jk} | \dots; M_{0,1}]$ (for $2 \leq j < k \leq 4$) and $[L_{012} | \dots; M_{0,l}]$ (for $l = 2, 3$) are degenerate. Here for fixed M_2 and M_3 we use L_3 as the special L -face. So we are in Case (ii) of §4.2.

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