Multiple polylogarithms: analytic continuation, monodromy, and variations of mixed Hodge structures

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Dedicated to the Memory of Kuo-Tsai Chen and Wei-Liang Chow

Abstract. In this exposition we shall describe a new way to analytically continue the multiple polylogarithms by using Chen's theory of iterated path integrals. Then we explicitly determine the good unipotent variations of mixed Hodge-Tate structures (MHS) related to multiple logarithms and some other multiple polylogarithms of lower weights. Following Deligne and Beilinson we define the single-valued real analytic version of the multiple polylogarithms which generalizes the well-known result of Zagier on classical polylogarithms. At the end, motivated by Zagier's conjecture we pose a problem which relates the special values of multiple Dedekind zeta functions of a number field to the single-valued version of multiple polylogarithms. The main results of this paper with complete proofs will appear elsewhere.

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1 Introduction

In recent years, there is a revival of interest in multi-valued classical polylogarithms and their single-valued cousins. In the mean time there have been a number of generalizations of these functions such as Grassmannian polylogs [21, 22, 25], Chow polylogs [16], elliptic polylogs [3, 28, 30, 36], p-adic polylogs [9], infinitesimal (p-adic) polylogs [6, 13], finite polylogs [5, 13, 29], and multiple polylogs [14, 18, 19, 20]. For any positive integer $m_1, \dots, m_n$, Goncharov [14] defines the multiple polylogs of complex variables as follows:

$$Li_{m_1,\dots,m_n}(x_1,\dots,x_n) = \sum_{0<k_1<k_2<\dots<k_n} \frac{x_1^{k_1} x_2^{k_2} \dots x_n^{k_n}}{k_1^{m_1} k_2^{m_2} \dots k_n^{m_n}}, \qquad |x_i| < 1. \tag{1}$$

Conventionally one refers n as the *depth* and $K := m_1 + \dots + m_n$ as the *weight*. When the depth $n = 1$ the function is nothing but the classical polylogarithm. More than a century ago it was already known to H. Poincaré [31] that hyperlogarithms

$$F_n\left(\begin{matrix} a_1,\dots,a_n \\ b_1,\dots,b_n \end{matrix}\Big| z\right) = \int_{b_n}^{z} \cdots \int_{b_2}^{t_3} \int_{b_1}^{t_2} \frac{dt_1}{t_1 - a_1} \frac{dt_2}{t_2 - a_2} \cdots \frac{dt_n}{t_n - a_n}$$

are important for solving differential equations. Notice that the multiple polylogarithm

$$Li_{m_1,\dots,m_n}(x_1,\dots,x_n) = (-1)^n F_K\left(\begin{matrix} a_1, & \overbrace{0,\dots,0}^{m_1-1 \text{ times}}, & \dots, & a_n, & \overbrace{0,\dots,0}^{m_n-1 \text{ times}} \\ 0\ , & 0,\dots,0\ , & \dots, & 0\ , & 0,\dots,0 \end{matrix} \,\Big|\, 1\right), \tag{2}$$

where $a_i = 1/(x_i \dots x_n)$ for $1 \le i \le n$. It is an iterated path integral in the sense of Chen [7] whose path lies in $\mathbb{C}$. One thus can easily enlarge its domain of definition to some open subset of $\mathbb{C}^n$. However, it is not obvious that this actually gives a genuine analytic continuation in the usual sense. It is one of our goals in this paper to analytically continue the above function to $\mathbb{C}^n$ as a multi-valued meromorphic function using Chen's iterated path integrals with paths all lie in $\mathbb{C}^n$. Note that even the classical polylogarithm is not holomorphic on $\mathbb{C}$, for example, $Li_1(x) = -\log(1-x)$ is clearly multi-valued.

In early 1980s Deligne [10] discovers that the dilogarithm gives rise to a good variation of mixed Hodge-Tate structures. This has been generalized to polylogarithms (cf. [23]) following Ramakrishnan's computation of the monodromy of the polylogarithms. The monodromy computation also yields the single-valued variant $\mathcal{L}_n(z)$ of the polylogarithms (cf. [1, 37]). These functions in turn have significant applications in arithmetic such as Zagier's conjecture [37, p.622]. On the other hand, as pointed out in [17, 19], "higher cyclotomy theory" should study the multiple polylogarithm motives at roots of unity, not only those of the polylogarithms. For this reason we want to look at the variations of mixed Hodge structures associated with the multiple polylogarithms and see how far we can generalize the classical results.

According to the theory of framed mixed Hodge-Tate structures the multiple polylogarithms are period functions of some variations of mixed Hodge-Tate structures (see [2], [14, §12] and [14, §3.5]). Wojtkowiak [35] studies mixed Hodge structures of iterated integrals over $\mathbb{CP}\setminus\{0,1,\infty\}$ and investigates functional equations arising from there. In this paper we adopt a down to earth approach different from [35] and compute *explicitly* the variations of mixed Hodge-Tate structures related to the multiple logarithms

$$\mathfrak{L}_n(x_1,\dots,x_n) := Li_{\underbrace{1,\dots,1}_{n \text{ times}}}(x_1,\dots,x_n).$$

The key step of this approach is our new definition of analytic continuation of the multiple polylogarithms using Chen's iterated path integrals over $\mathbb{CP}^n \setminus D_n$ with some non-normal crossing divisor D_n. In order to have "reasonable" variations we should be able to control their behavior at "infinity" D_n. This requires us to deal with the natural extension of the variations to the infinity using the classical result of Deligne [11, Proposition 5.2]. By the same idea we are able to treat all the weight three multiple polylogarithms and present a result for the double polylogarithms. We observe that the old form (2) of polylogarithms is *not* suitable for the investigation of the MHS at the infinity.

As an important application of the above explicit computation, in the last section of this paper we describe an approach to computing the single-valued real analytic version of the multiple polylogarithms following an idea of Beilinson and Deligne [1].

For example, we find the single-valued real analytic double logarithm

$$\begin{aligned}\mathcal{L}_{1,1}(x,y) =& \operatorname{Im}\big(Li_{1,1}(x,y)\big) - \arg(1-y)\log|1-x| - \arg(1-xy)\log\Big|\frac{x(1-y)}{x-1}\Big| \\ =& \mathcal{L}_2\Big(\frac{xy-y}{1-y}\Big) - \mathcal{L}_2\Big(\frac{y}{y-1}\Big) - \mathcal{L}_2(xy)\end{aligned}$$

where $\mathcal{L}_2(z)$ is the famous single-valued dilogarithm. Similar identities in weight three case are also listed. It is a remarkable phenomenon that in all these identities no terms of lower weight occurs which is drastically different from the classical situation.

The motivation of this paper comes from [17, §2,3] where the Hodge-Tate structures coming from the double logarithms are discussed, and from [1] where an elegant construction of the single-valued real analytic version of classical polylogarithms are given. The author wishes to thank his advisor Sasha Goncharov for his encouragement and generous help and R. Hain for answering some of my questions concerning the good unipotent variations of mixed Hodge structures. H. Gangl kindly informed the author of the preprint [35] of Wojtkowiak in which conjectures generalizing Zagier's are also considered. The author also thanks the referee whose comments and suggestions make the exposition clearer.

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2 Analytic continuation of multiple polylogarithms

In this section we define the analytic continuation of the multiple polylogarithms different from (2) by using Chen's theory of iterated path integrals.

2.1 Chen's theory of iterated path integrals.

The primary references of this subsection are two of Chen's papers [7] and [8].

For a 1-form $f(t)dt$ over $\mathbb{R}$ the integral $\int_a^b f(t)dt$ is understood in the usual way. For $r > 1$, define inductively

$$\int_a^b f_1(t)dt \cdots f_r(t)dt = \int_a^b \left(\int_a^t f_1(\tau)d\tau \cdots f_{r-1}(\tau)d\tau \right) f_r(t)dt.$$

When $r = 0$, set the integral to be 1. For example, the classical polylogarithm

$$Li_n(x) = \int_0^x \frac{dt}{1-t} \underbrace{\frac{dt}{t} \cdots \frac{dt}{t}}_{(n-1)\text{ times}}.$$

More generally, let $w_1, w_2, \dots$ be 1-forms on a manifold M and let $\alpha : [0,1] \to M$ be a piecewise smooth path. Write

$$\alpha^* w_i = f_i(t)dt$$

and define the *iterated path integral*

$$\int_\alpha w_1 \cdots w_r = \int_0^1 f_1(t)dt \cdots f_r(t)dt.$$

Here we follow Chen's original convention which is different from the one adopted by Deligne [12] who writes $\int_\alpha w_r \cdots w_1$ to mean our $\int_\alpha w_1 \cdots w_r$.

The following results are crucial for the application of the Chen's theory of iterated path integrals.

Lemma 2.1. *Let w_i $(i \geq 1)$ be $\mathbb{C}$-valued 1-forms on a manifold M.*

(i) The value of $\int_\alpha w_1 \cdots w_r$ is independent of the parameterization of α.

(ii) If $\alpha, \beta : [0,1] \longrightarrow M$ are composable paths (i.e. $\alpha(1) = \beta(0)$ or α followed by β), then

$$\int_{\alpha\beta} w_1 \cdots w_r = \sum_{j=0}^{r} \int_\alpha w_1 \cdots w_i \int_\beta w_{i+1} \cdots w_r.$$

Here, we set $\int_\alpha \phi_1 \cdots \phi_m = 1$ if $m = 0$.

(ii) For every path α,

$$\int_{\alpha^{-1}} w_1 \cdots w_r = (-1)^r \int_\alpha w_r \cdots w_1.$$

(iv) For every path α,

$$\int_\alpha w_1 \cdots w_r \int_\alpha w_{r+1} \cdots w_{r+s} = \sum_\sigma \int_\alpha w_{\sigma(1)} \cdots w_{\sigma(r+s)},$$

where σ ranges over all shuffles of type (r,s), i.e., permutations σ of $r+s$ letters with $\sigma^{-1}(1) < \cdots < \sigma^{-1}(r)$ and $\sigma^{-1}(r+1) < \cdots < \sigma^{-1}(r+s)$.

Proof. (i) can be derived from the theorem on [7, p. 361]. (ii) and (iii) are formulas (1.6.1) and (1.6.2) of [7] respectively. Ree [33] discovered the shuffle relation (iv) which appeared as (1.5.1) in [7]. □

Lemma 2.2. *If $w_i^{(j)}$ are closed 1-forms for $1 \leq i \leq r$ and $1 \leq j \leq n$ such that $\sum_j w_1^{(j)} \wedge w_2^{(j)} = \sum_j w_2^{(j)} \wedge w_3^{(j)} = \cdots = \sum_j w_{r-1}^{(j)} \wedge w_r^{(j)} = 0$ then $\sum_j \int_\alpha w_1^{(j)} w_2^{(j)} \cdots w_r^{(j)}$ only depends on the homotopy class of α.*

Proof. The case $j = 1$ is proved on [7, p. 366]. The case $r = 2$ can be found on [7, p. 368]. The general case follows from a similar argument. □

2.2 The index set $\mathfrak{S}(m_1, \ldots, m_n)$

Our analytic continuation of the multiple polylogarithms of depth n will be produced by some Chen's iterated path integrals over $\mathbb{C}^n$. To write down the formula explicitly we need to introduce an index set with two different kind of orderings.

2.2.21. Definition. Define the index set

$$\mathfrak{S}(m_1,\dots,m_n)=\{\mathbf{i}=(i_1,\dots,i_n):\ 0\le i_t\le m_t \text{ for } t=1,\cdots,n\}$$

and the weight function $|\cdot|$ on $\mathfrak{S}(m_1,\dots,m_n)$ by

$$|(i_1,\dots,i_n)|=i_1+\cdots+i_n.$$

For brevity, we write $\mathbf{0}=(0,\dots,0)\in\mathfrak{S}(m_1,\dots,m_n)$ which is the only index of weight 0 in $\mathfrak{S}(m_1,\dots,m_n)$ and $\mathbf{1}_K=(m_1,\dots,m_n)\in\mathfrak{S}(m_1,\dots,m_n)$ which is the only index of the highest weight $K:=m_1+\cdots+m_n$ in $\mathfrak{S}(m_1,\dots,m_n)$. We also define the depth function of the index $(i_1,\dots,i_n)$ by $\sharp\{t:\ i_t\ne 0\}$, i.e., the number of nonzero components.

2.2.2. A complete ordering. The complete ordering is defined as follows. Let $\mathbf{i}=(i_1,\dots,i_n)$ and $\mathbf{j}=(j_1,\dots,j_n)$. If $|\mathbf{i}|<|\mathbf{j}|$ then $\mathbf{i}<\mathbf{j}$ (or, equivalently, $\mathbf{j}>\mathbf{i}$). If $|\mathbf{i}|=|\mathbf{j}|$ then $\mathbf{i}>\mathbf{j}$ if $\max\{i_t:1\le t\le n\}>\max\{j_t:1\le t\le n\}$. Otherwise, we compare the second largest components of $\mathbf{i}$ and $\mathbf{j}$, and so on. If $\{i_1,\dots,\imath_n\}=\{j_1,\dots,j_n\}$ as two sets then the usual lexicographic order from left to right is in force with $0<1<\cdots$. For instance, $(0,0,1)<(1,0,1)<(1,1,0)<(0,2,0)$ in $\mathfrak{S}(1,2,1)$.

Remark 2.3. In the multiple logarithm case, namely, when $m_1=\cdots=m_n=1$, there is a one-to-one correspondence between $\mathfrak{S}_n$ and the set of non-negative integers less than 2^n. Thus one is tempted to use the conventional order of positive integers in binary forms. However this is not suitable in our situation.

2.2.3. A partial ordering and the retraction map. The partial ordering is defined as follows. Let $\mathbf{i}=(i_1,\dots,i_n)$ and $\mathbf{j}=(j_1,\dots,j_n)$. We set $\mathbf{j}\prec\mathbf{i}$ (or, equivalently, $\mathbf{i}\succ\mathbf{j}$) if $j_t\le i_t$ for every $1\le t\le n$. For example $(0,0,1,0)\prec(0,1,1,0)$ in $\mathfrak{S}(1,1,1,1)$ but $(1,0,0,0)\not\prec(0,1,1,0)$ and $(1,0,0,0)\not\succ(0,1,1,0)$. Clearly $\mathbf{j}\prec\mathbf{i}$ implies $\mathbf{j}<\mathbf{i}$ but not vice versa.

Suppose $\mathbf{i}$ has depth k with $i_{\tau_s}\ne 0$ for $1\le s\le k$ while $\mathbf{j}$ has depth l and $j_{t_r}\ne 0$ for $1\le r\le l$. If $\mathbf{j}\prec\mathbf{i}$ then we can write $t_r=\tau_{\alpha_r}$ for $1\le r\le l$. For such $\mathbf{i}$ and $\mathbf{j}$ we define the $\mathbf{i}$-th retraction map $\rho_{\mathbf{i}}$ from $\mathfrak{S}(m_1,\dots,m_n)$ to $\mathfrak{S}(i_{\tau_1},\dots,i_{\tau_k})$ as follows. The entry of $\rho_{\mathbf{i}}(\mathbf{j})$ is $j_{\tau_{\alpha_r}}$ if it is at the α_r-th $(1\le r\le l)$ component and 0 at all other components. For instance $\rho_{(02010)}(01000)=(10)\in\mathfrak{S}(2,1)$. In particular, $\rho_{\mathbf{i}}(\mathbf{i})=(i_{\tau_1},\dots,i_{\tau_k})$ has highest weight in $\mathfrak{S}(i_{\tau_1},\dots,i_{\tau_k})$.

2.2.4. Vector indices. Let $\mathfrak{S}^K(m_1,\dots,m_n)$ be the set of K-tuples $\overrightarrow{\mathbf{j}}=(\mathbf{j}_1,\dots,\mathbf{j}_K)$ of $\mathfrak{S}(m_1,\dots,m_n)$ such that $|\mathbf{j}_t|=t$ and $\mathbf{j}_1\prec\cdots\prec\mathbf{j}_K=\mathbf{1}_K$. One may think $\overrightarrow{\mathbf{j}}$ as a length K queue of indices of $\mathfrak{S}(m_1,\dots,m_n)$ in which each index is produced by increasing some component of the preceding index by 1.

2.2.5. Additional notation. We fix $\mathbf{u}_s:=(0,\dots,0,1,0,\dots,0)\in\mathfrak{S}(m_1,\dots,m_n)$ of weight 1 throughout the exposition, where the entry 1 is at the s-th component. Whenever the s-th component i_s of $\mathbf{i}$ satisfies $i_s<m_s$ we can increase i_s by 1 to get a new index which is denoted by $\mathbf{i}+\mathbf{u}_s$. If $i_s>0$ we similarly define $\mathbf{i}-\mathbf{u}_s$ as the index with the s-th component of $\mathbf{i}$ decreased by 1. Fix $\mathbf{v}_s=\mathbf{1}_K-m_s\mathbf{u}_s\in\mathfrak{S}(m_1,\dots,m_n)$ whose components are nonzero except at the s-th position.

When $m_1 = \cdots = m_n = 1$ we write $\mathfrak{S}(1,\dots,1) = \mathfrak{S}_n$.

2.2.6. Transposition functions. Let $\overrightarrow{\mathbf{j}} = (\mathbf{j}_1,\dots,\mathbf{j}_K) \in \mathfrak{S}^K(m_1,\dots,m_n)$. For $1 < r \le K$ we write

$$\mathbf{j}_r = \mathbf{j}_{r-1} + \mathbf{u}_s = (t_1,\dots,t_s,0,\dots,0,t_a,\dots,t_n), \quad 0 \le s < a \le n+1, \ t_a \neq 0.$$

Here if $a = n+1$ then the last nonzero component of $\mathbf{j}_r$ is t_s. We define the transposition functions on $\mathbf{i} = (i_1,\dots,i_n) \in \mathfrak{S}(m,\dots,m)$ with $m = \max\{m_1,\dots,m_n\}$ by

$$T_0^r = \mathrm{id}, \quad T_1^r(\mathbf{i}) = (i_{\sigma(1)},\dots,i_{\sigma(n)}),$$

where if $t_s > 1$ or $a = n+1$ then $\sigma = \mathrm{id}$ whereas if $t_s = 1$ and $a \le n$ then σ is the transposition in the symmetric group of n elements that exchanges s and a.

2.3 Analytic continuation of multiple polylogarithms

In this section we define the analytic continuation of the multiple polylogarithms which will be used to calculate the monodromy of multiple polylogarithms.

Let $\mathbf{x} = (x_1,\dots,x_n)$ be a variable over $\mathbb{C}^n$. Define $S_n = \mathbb{C}^n \setminus X_n$ where the divisor

$$X_n = \Big\{\mathbf{x} \in \mathbb{C}^n : \prod_{1\le i\le n} x_j(1-x_j) \prod_{1\le j<k\le n} \big(1 - x_j \dots x_k\big) = 0\Big\}. \tag{3}$$

Set the domain

$$D_n = \Big\{(x_1,\dots,x_n) \in \mathbb{C}^n : \Big|x_j - \frac{1}{2}\Big| < \frac{1}{2}, j = 1,\dots,n\Big\} \subset S_n.$$

Suppose the depth of $\mathbf{i} = (i_1,\dots,i_n)$ is k and $i_{\tau_1} \neq 0,\dots,i_{\tau_k} \neq 0$. We define

$$a_t = a_t(\mathbf{x}) := (x_t \dots x_n)^{-1}, \quad 1 \le t \le n,$$

and

$$\mathbf{x}(\mathbf{i}) = \mathbf{y} = (y_1,\dots,y_k), \qquad y_m = \prod_{\alpha=\tau_m}^{\tau_{m+1}-1} x_\alpha = \frac{a_{\tau_{m+1}}(\mathbf{x})}{a_{\tau_m}(\mathbf{x})}, \quad 1 \le m \le k, \tag{4}$$

with $\tau_{k+1} = n+1$ and $a_{n+1} = 1$. We also write $a_m(\mathbf{y}) = (y_m \cdots y_k)^{-1} = a_{\tau_m}(\mathbf{x})$. Note that $\mathbf{x}(\mathbf{i}) \in \mathbb{C}^k$ which is the reason why we call k the depth of $\mathbf{i}$.

We begin with some 1-forms which will be used to express the multiple polylogarithms. Take $\overrightarrow{\mathbf{j}} \in \mathfrak{S}^K(m_1,\dots,m_n)$ and $\mathbf{j}_r = \mathbf{j}_{r-1} + \mathbf{u}_s$ as given in §2.2.6. For any $(\delta_1,\dots,\delta_K) \in \mathfrak{S}_K$, namely, $\delta_t = 0$ or 1, let $\mathbf{y} = \mathbf{x}$ if $r = K$ and

$$\mathbf{y} = (y_1,\dots,y_l) = \mathbf{x}\big(T_{\delta_{r+1}}^{r+1} \circ \cdots \circ T_{\delta_K}^K(\mathbf{j}_r)\big) \qquad \text{if} \quad 1 \le r < K,$$

where l is the depth of $\mathbf{j}_r$ because the transposition functions do not change the depth of an index. We let $t_{\alpha_1} \neq 0,\dots,t_{\alpha_l} \neq 0$ and $s = \alpha_\lambda$ (because $t_s \neq 0$) and set

$$w_{\overrightarrow{\mathbf{j}}}^{r,\delta_r}(\mathbf{y}) := \begin{cases} 0 & \text{if } t_s > 1 \text{ and } \delta_r = 1, \\ dy_\lambda/y_\lambda & \text{if } t_s > 1 \text{ and } \delta_r = 0, \\ dy_\lambda/(1-y_\lambda) & \text{if } t_s = 1 \text{ and } \delta_r = 0, \\ dy_\lambda/y_\lambda(y_\lambda - 1) & \text{if } \lambda < l,\ t_s = 1 \text{ and } \delta_r = 1, \\ 0 & \text{if } \lambda = l,\ t_s = 1 \text{ and } \delta_r = 1. \end{cases}$$

Obviously $w^{r,\delta_r}_{\overrightarrow{\mathbf{J}}}(\mathbf{y})$ is always a closed 1-form whose singularities lie only along X_n.

Proposition 2.4. *Let $\mathbf{0}_n = \mathbf{0}$ be the origin in $\mathbb{C}^n$. Let $\int_p \bigsqcup_{r=1}^K w_r$ denote the iterated integral $\int_p w_1 \cdots w_K$. Then for every $\mathbf{x} \in D_n$*

$$Li_{m_1,\dots,m_n}(\mathbf{x}) = \int_{\mathbf{0}}^{\mathbf{x}} \sum_{\substack{(\delta_1,\dots,\delta_K)\in\mathfrak{S}_K \\ \overrightarrow{\mathbf{J}}\in\mathfrak{S}^K(m_1,\dots,m_n)}} \bigsqcup_{r=1}^{K} w^{r,\delta_r}_{\overrightarrow{\mathbf{J}}} \left(\mathbf{x}\Big(T^{r+1}_{\delta_{r+1}} \circ \cdots \circ T^K_{\delta_K}(\mathbf{j}_r)\Big)\right),$$

where the paths of the iterated integral lie entirely in D_n

Proof. This is proved by induction on K by using the power series expansion (1) to express the derivative of $Li_{m_1,\dots,m_n}(\mathbf{x})$ in terms of polylogarithms of weight $K-1$. □

By the above proposition we can now define the analytic continuation of $Li_{m_1,\dots,m_n}(\mathbf{x})$ to S_n as the iterated path integral

$$Li_{m_1,\dots,m_n}(\mathbf{x}) = \int_{\mathbf{0}_n}^{\mathbf{x}} \sum_{\substack{(\delta_1,\dots,\delta_K)\in\mathfrak{S}_K \\ \overrightarrow{\mathbf{J}}\in\mathfrak{S}^K(m_1,\dots,m_n)}} \bigsqcup_{r=1}^{K} w^{r,\delta_r}_{\overrightarrow{\mathbf{J}}} \left(\mathbf{x}\Big(T^{r+1}_{\delta_{r+1}} \circ \cdots \circ T^K_{\delta_K}(\mathbf{j}_r)\Big)\right). \qquad (5)$$

where all the paths lie inside S_n. Note that all the 1-forms appearing in (5) are rational forms with logarithmic singularities along X_n.

Example 2.5. When $n = 1$,

$$Li_1(x) = \int_0^x d\log\Big(\frac{1}{1-x}\Big) = -\log(1-x).$$

When $n = 2$, $\mathfrak{S}_2 = \{(0,0),(0,1),(1,0),(1,1)\}$ and there are two elements in $\mathfrak{S}^2$: $((0,1),(1,1))$ and $((1,0),(1,1))$. Let $\mathbf{x} = (x,y)$ then

$$\mathbf{x}(0,1) = y, \quad \mathbf{x}(1,0) = xy, \quad \mathbf{x}(1,1) = (x,y).$$

Thus

$$\begin{aligned} Li_{1,1}(x,y) &= \int_{\mathbf{0}}^{\mathbf{x}} w_1(\mathbf{x}(0,1))w_1(\mathbf{x}) + w_1(\mathbf{x}(1,0))w_2(\mathbf{x}) \\ &= \int_{(0,0)}^{(x,y)} \frac{dy}{1-y}\frac{dx}{1-x} + \frac{d(xy)}{1-xy}\left(\frac{dy}{1-y} + \frac{dx}{x(x-1)}\right). \end{aligned}$$

Lemma 2.6. *The iterated path integral of* (5) *depends only on the homotopy class of the path from $\mathbf{0}$ to $\mathbf{x}$.*

Proof. By induction on the weight K and Lemma 2.2. □

2.4 Multiple logarithms

We now specialize to the multiple logarithm cases where $m_1 = \cdots = m_n = 1$. Then we have $\mathfrak{S}(1,\dots,1) = \mathfrak{S}_n$ and $K = n$. Though we can get the analytic continuation of the multiple logarithms by (5) immediately, we want to derive a cleaner expression in this special case.

For any $\mathbf{i} = (i_1,\dots,i_n) \in \mathfrak{S}_n$ with $i_s = 0$ we define

$$\mathrm{pos}(\mathbf{i}, \mathbf{i}+\mathbf{u}_s) = s$$

as the position where the component is increased by 1. For example $\mathrm{pos}\big((1,0),(1,1)\big) = 2$. We define the position functions $f_n^1,\dots,f_n^n$ on $\overrightarrow{\mathbf{J}} \in \mathfrak{S}_n^n$ as follows:

$$f_n^1(\overrightarrow{\mathbf{J}}) = 1, \quad f_n^t(\overrightarrow{\mathbf{J}}) = \mathrm{pos}\big(\mathbf{j}_{t-1}, \mathbf{j}_t\big), \text{ for } 2 \le t \le n.$$

These functions tell us the places where the increments occur in the queue of $\overrightarrow{\mathbf{J}}$.

The following closed 1-forms will play important roles in the rest of this note:

$$w_1(\mathbf{x}) := d\log\Big(\frac{1}{1-x_1}\Big); \quad w_t(\mathbf{x}) := d\log\Big(\frac{1-x_{t-1}^{-1}}{1-x_t}\Big), \quad 2 \le t \le n.$$

Proposition 2.7. *The multiple logarithm $\mathfrak{L}_n(\mathbf{x})$ is a multi-valued meromorphic function on $\mathbb{C}^n$. For $\mathbf{x} \in S_n$ one has*

$$\mathfrak{L}_n(\mathbf{x}) = \sum_{\overrightarrow{\mathbf{J}}=(\mathbf{j}_1,\dots,\mathbf{j}_n)\in\mathfrak{S}_n^n} \int_{\mathbf{0}}^{\mathbf{x}} w_{f_n^1(\overrightarrow{\mathbf{J}})}(\mathbf{x}(\mathbf{j}_1))w_{f_n^2(\overrightarrow{\mathbf{J}})}(\mathbf{x}(\mathbf{j}_2))\cdots w_{f_n^n(\overrightarrow{\mathbf{J}})}(\mathbf{x}(\mathbf{j}_n)). \tag{6}$$

3 Multiple logarithm variations of MHS

In this section we will define the variation matrix $\mathcal{M}_{[n]}(\mathbf{x})$ coming from the multiple logarithms of depths up to n. We will show that it is a $2^n \times 2^n$ multi-valued matrix which defines a good variation of a MHS over $S_n = \mathbb{C}^n \setminus X_n$ where X_n is the divisor defined by (3).

Remark 3.1. In fact, the irreducible component $x_n = 0$ in X_n is not needed in the case of multiple logarithms. But the variation matrix corresponding to general multiple polylogarithms may have singularities along this component, for example, $\mathcal{M}_{1,2}(x_1,x_2)$ of the double polylogarithm $Li_{1,2}(x_1,x_2)$. See §5.

3.1 The variation matrix

The double logarithm was treated in [17, §2] by Goncharov. We rewrite his $A_{1,1}(x,y)$ as $\mathcal{M}_{1,1}(x,y)$ and try to generalize this to arbitrary multiple logarithm variation matrix $\mathcal{M}_{[n]}(\mathbf{x})$ for $\mathbf{x} \in S_n$. Observe that on the index set $\mathfrak{S}_n$ the depth and the weight functions coincide.

Definition 3.2. Suppose $|\mathbf{i}| = k$ and $i_{\tau_1} = \cdots = i_{\tau_k} = 1$. Suppose $|\mathbf{j}| = l$ and $j_{t_1} = \cdots = j_{t_l} = 1$.

(1) If $\mathbf{j} \not\prec \mathbf{i}$, we define the $(\mathbf{i}, \mathbf{j})$-th entry of $\mathcal{M}_{[n]}(\mathbf{x})$ to be 0.

(2) If $\mathbf{j} \prec \mathbf{i}$ then we let $t_r = \tau_{\alpha_r}$ for $1 \le r \le l$. Set $t_0 = \alpha_0 = 0$, $t_{l+1} = n+1$, $\alpha_{l+1} = k+1$. Define the $(\mathbf{i}, \mathbf{j})$-th entry of $\mathcal{M}_{[n]}(\mathbf{x})$ as $(2\pi i)^l E_{\mathbf{i},\mathbf{j}}(\mathbf{x})$ where

$$E_{\mathbf{i},\mathbf{j}}(\mathbf{x}) = \gamma^k_{\rho_{\mathbf{i}}(\mathbf{j})}(\mathbf{x}(\mathbf{i})) := \mathfrak{L}_{\alpha_1-1}\big(x_{\tau_1}\cdots x_{\tau_2-1}, x_{\tau_2}\cdots x_{\tau_3-1}, \cdots, x_{\tau_{\alpha_1-1}}\cdots x_{t_1-1}\big)\cdot$$
$$\cdot \prod_{r=1}^{l} \mathfrak{L}_{\alpha_{r+1}-\alpha_r-1}\left(\frac{1 - x_{t_r}\cdots x_{\tau_{\alpha_r+2}-1}}{1 - x_{t_r}\cdots x_{\tau_{\alpha_r+1}-1}}, \cdots, \frac{1 - x_{t_r}\cdots x_{t_{r+1}-1}}{1 - x_{t_r}\cdots x_{\tau_{\alpha_{r+1}-1}-1}}\right). \quad (7)$$

Here $\mathfrak{L}_0 = 1$.

Examples 3.3. On the last row of $\mathcal{M}_{[n]}(\mathbf{x})$ one has

$$E_{\mathbf{1},\mathbf{j}}(\mathbf{x}) = \prod_{r=0}^{l} \mathfrak{L}_{t_{r+1}-t_r-1}\Big(\frac{1 - x_{t_r}x_{t_r+1}}{1 - x_{t_r}}, \cdots, \frac{1 - x_{t_r}\cdots x_{t_{r+1}-1}}{1 - x_{t_r}\cdots x_{t_{r+1}-2}}\Big). \quad (8)$$

In particular, $E_{\mathbf{1},\mathbf{0}} = \gamma^n_{\mathbf{0}}(\mathbf{x}) = \mathfrak{L}_n(\mathbf{x})$ and $E_{\mathbf{1},\mathbf{1}} = \gamma^n_{\mathbf{1}}(\mathbf{x}) = 1$. Another interesting case is when $|\mathbf{i}| = l$, $|\mathbf{j}| = l-1$ and $\mathbf{j} \prec \mathbf{i}$. Suppose $i_{t_1} = \cdots = i_{t_l} = 1$ and $j_{t_s} = 0$ then $E_{\mathbf{i},\mathbf{j}}(\mathbf{x}) = -\log f_{\mathbf{i},\mathbf{j}}(\mathbf{x})$ where

$$f_{\mathbf{i},\mathbf{j}}(\mathbf{x}) = \begin{cases} 1 - x_1 \dots x_{t_1} & \text{if } s = 1, \\ \dfrac{x_{t_{s-1}} \dots x_{t_s-1}(x_{t_s} \dots x_{t_{s+1}-1} - 1)}{1 - x_{t_{s-1}} \dots x_{t_s-1}} & \text{if } s \ge 2. \end{cases} \quad (9)$$

We now fix a standard basis $\{e_{\mathbf{i}} : \mathbf{i} \in \mathfrak{S}_n\}$ of $\mathbb{C}^{2^n}$ consisting of column vectors. Suppose $|\mathbf{i}| = k$. It follows from definition that the $\mathbf{i}$-th row of $\mathcal{M}_{[n]}(\mathbf{x})$ is

$$R_{\mathbf{i}} := \sum_{\mathbf{j}\prec\mathbf{i}} (2\pi i)^{|\mathbf{j}|}\gamma^k_{\rho_{\mathbf{i}}(\mathbf{j})}\big(\mathbf{x}(\mathbf{i})\big)e^T_{\mathbf{j}} = (2\pi i)^k e^T_{\mathbf{i}} + \sum_{\mathbf{j}\precneqq\mathbf{i}} (2\pi i)^{|\mathbf{j}|}\gamma^k_{\rho_{\mathbf{i}}(\mathbf{j})}\big(\mathbf{x}(\mathbf{i})\big)e^T_{\mathbf{j}}, \quad (10)$$

where $e^T_{\mathbf{j}}$ are now row vectors. Note that $\gamma^k_{\rho_{\mathbf{i}}(\mathbf{i})} = \gamma^k_{\mathbf{1}_k} = 1$ by definition. It is clear that the first entry (i.e., $\mathbf{j} = \mathbf{0}$) of this row is $\mathfrak{L}_k\big(\mathbf{x}(\mathbf{i})\big)$.

Let us call the minor of $\mathcal{M}_{[n]}(\mathbf{x})$ consisting of rows beginning with k-tuple logarithms *the k-th block*. It has $\binom{n}{k}$ rows with row indices $|\mathbf{i}| = k$.

Lemma 3.4. *The matrix $\mathcal{M}_{[n]}(\mathbf{x})$ is a lower triangular matrix. Moreover, the columns with $|\mathbf{j}| = k$ of the k-th block of $\mathcal{M}_{[n]}(\mathbf{x})$ is $(2\pi i)^k$ times the identity matrix of rank $\binom{n}{k}$.*

Proof. The lemma follows directly from equation (10) because if $\mathbf{j} \precneqq \mathbf{i}$ then $\mathbf{j} < \mathbf{i}$. □

Lemma 3.5. *The $\mathbf{j}$-th column of $\mathcal{M}_{[n]}(\mathbf{x})$ is*

$$(2\pi i)^{|\mathbf{j}|}C_{\mathbf{j}} = (2\pi i)^{|\mathbf{j}|} \sum_{\mathbf{i}\succ\mathbf{j}} \gamma^{|\mathbf{i}|}_{\rho_{\mathbf{i}}(\mathbf{j})}\big(\mathbf{x}(\mathbf{i})\big)e_{\mathbf{i}},$$

where $\mathbf{x}(\mathbf{i})$ are defined by equation (4) depending on $\mathbf{i}$.

Proof. Use equation (10). □

Example 3.6. Let $\mathfrak{L}_0 = 1$. By definition or the above proposition the first column

$$C_{\mathbf{0}}(\mathbf{x}) = \left[\mathfrak{L}_{|\mathbf{i}|}(\mathbf{x}(\mathbf{i})) : \ \mathbf{i} \in \mathfrak{S}_n\right]^T.$$

Proposition 3.7. *The columns of $\mathcal{M}_{[n]}(\mathbf{x})$ form the set of the fundamental solutions of the system of differential equations*

$$\begin{cases} d\,X_{\mathbf{0}} = 0, \\ d\,X_{\mathbf{i}} = \displaystyle\sum_{|\mathbf{k}|=|\mathbf{i}|-1,\, \mathbf{k}\prec\mathbf{i}} -X_{\mathbf{k}}\, d\log f_{\mathbf{i},\mathbf{k}}(\mathbf{x}) \quad \text{for all } 1 \le |\mathbf{i}| \le n \end{cases} \tag{11}$$

where $f_{\mathbf{i},\mathbf{k}}(\mathbf{x})$ are rational functions defined by (9).

3.2 Monodromy of $\mathcal{M}_{[n]}(\mathbf{x})$

Fix an embedding $\mathbb{C}^n \hookrightarrow \mathbb{CP}^n$. Let $\mathcal{X}_n = X_n \cup (\mathbb{CP}^n \setminus \mathbb{C}^n)$. Let $M_r(\mathbb{C})$ be the set of $r \times r$ matrices over $\mathbb{C}$. Put

$$\boldsymbol{\omega} = \Big(c_{\mathbf{i},\mathbf{j}}\Big)_{\mathbf{i},\mathbf{j}\in\mathfrak{S}_n} \in H^0(\mathbb{CP}^n, \Omega^1_{\mathbb{CP}^n}(\log(\mathcal{X}_n))) \otimes M_{2^n}(\mathbb{C}), \tag{12}$$

where

$$c_{\mathbf{i},\mathbf{j}} = \begin{cases} -d\log f_{\mathbf{i},\mathbf{j}}(\mathbf{x}) & \text{if } |\mathbf{j}| = |\mathbf{i}| - 1,\ \mathbf{j} \prec \mathbf{i}, \\ 0 & \text{otherwise.} \end{cases}$$

All of the 1-forms in $\boldsymbol{\omega}$ have logarithmic singularity on $\mathcal{X}_n$ because $f_{\mathbf{i},\mathbf{j}}(\mathbf{x})$ are all rational functions given by (9).

Example 3.8. When $n = 2$ we have

$$\boldsymbol{\omega} = \begin{bmatrix} 0 & & & \\ -d\log(1-y) & 0 & & \\ -d\log(1-xy) & & 0 & \\ 0 & -d\log(1-x) & -d\log\frac{x(1-y)}{x-1} & 0 \end{bmatrix}.$$

We have seen in Proposition 3.7 that $\mathcal{M}_{[n]}(\mathbf{x})$ is a fundamental solution of the first order linear partial differential equation

$$d\Lambda = \boldsymbol{\omega}\Lambda, \tag{13}$$

where Λ is a possibly multi-valued function $S_n \longrightarrow M_{2^n}(\mathbb{C})$. Moreover $\mathcal{M}_{[n]}(\mathbf{x})$ is a unipotent matrix for very $\mathbf{x} \in S_n$. Applying d on equation (13) and plugging in $\Lambda = \mathcal{M}_{[n]}(\mathbf{x})$ we get

$$0 = d\boldsymbol{\omega}\mathcal{M}_{[n]}(\mathbf{x}) - \boldsymbol{\omega} \wedge d\mathcal{M}_{[n]}(\mathbf{x}) = (d\boldsymbol{\omega} - \boldsymbol{\omega} \wedge \boldsymbol{\omega})\mathcal{M}_{[n]}(\mathbf{x})$$

Because $\mathcal{M}_{[n]}(\mathbf{x})$ is invertible and $\boldsymbol{\omega}$ is closed we get

$$d\boldsymbol{\omega} = 0, \quad \boldsymbol{\omega} \wedge \boldsymbol{\omega} = 0. \tag{14}$$

This shows that $\boldsymbol{\omega}$ is integrable.

The main goal of this section is to show that if we analytically continue every integral entry of $\mathcal{M}_{[n]}(\mathbf{x})$ along a common loop $q \in \pi_1(S_n, \mathbf{x})$, the resulting matrix will still be a fundamental solution $\mathcal{M}_{[n]}(\mathbf{x})M(q)$ of (13) where $M(q) \in \mathrm{GL}_{2^n}(\mathbb{Z})$. In what follows we also denote this action of q by $\Theta(q)$ operating on the left. We then define the monodromy representation

$$\begin{aligned} \rho_{\mathbf{x}} : \pi_1(S_n, \mathbf{x}) &\longrightarrow \mathrm{GL}_{2^n}(\mathbb{Z}) \\ q &\longmapsto M(q)^T. \end{aligned}$$

Here we take the transpose to ensure $\rho_{\mathbf{x}}$ to be a homomorphism because $M(pq) = M(q)M(p)$ by our convention. From the explicit computation in Theorem 3.12 we will see that $\rho_{\mathbf{x}}$ is a unipotent representation. By the definition in (7) in order to determine the monodromy of $\mathcal{M}_{[n]}(\mathbf{x})$ it is imperative that we resolve the monodromy of the multiple logarithms $\mathfrak{L}_n(\mathbf{x})$ first. The next three lemma can be found by induction using integral computations.

Lemma 3.9. *Let $1 \le s \le n$. Let p be a path from $\mathbf{0}$ to $\mathbf{x}$ in S_n. Let $q_s \in \pi_1(S_n, \mathbf{x})$ enclose the component $\mathcal{D}_{sn} = \{x_s \cdots x_n = 1\}$ only once in S_n but no other irreducible components of X_n such that $\int_{q_s} d\log(1 - x_s \cdots x_n) = -2\pi i$. Then*

$$(\Theta(q_s) - \mathrm{id})\mathfrak{L}_n(\mathbf{x}) = -2\pi i \mathfrak{L}_{s-1}(x_1, \dots, x_{s-1}) \cdot \mathfrak{L}_{n-s}(\mathbf{y}(s)),$$

where

$$\mathbf{y}(s) = \Big(\frac{1 - x_s x_{s+1}}{1 - x_s}, \dots, \frac{1 - x_s \dots x_n}{1 - x_s \dots x_{n-1}}\Big).$$

Lemma 3.10. *The monodromy of $\mathfrak{L}_n(\mathbf{x})$ about $\mathcal{D}_{ii} = \{x_i = 1\}$, $1 \le i < n$, and $\mathcal{D}_{ij} = \{x_i \cdots x_j = 1\}$, $1 \le i < j < n$, is trivial.*

Lemma 3.11. *Let $n > 1$. For any $1 \le a < b \le n$ set $F_{aa} = 1$ and*

$$F_{ab}(\mathbf{x}) = \mathfrak{L}_{b-a}\left(\frac{1 - x_a x_{a+1}}{1 - x_a}, \cdots, \frac{1 - x_a \cdots x_b}{1 - x_a \cdots x_{b-1}}\right).$$

Let $1 \le j < n$ and $q_{j0} \in \pi_1(S_n, \mathbf{x})$ (resp. $1 \le j < n$ and q_{1j}, $2 \le j \le n$ and q_{jn}) be a loop turning around the component $\mathcal{D}_{j0} = \{x_j = 0\}$ (resp. $\mathcal{D}_{1j} = \{x_1 \cdots x_j = 1\}$, resp. $\mathcal{D}_{jn} = \{x_j \cdots x_n = 1\}$), only once but no other irreducible components of X_n such that $\int_{q_{j0}} dx_j/x_j = 2\pi i$ (resp. $\int_{q_{1j}} d\log(1 - x_1 \cdots x_j) = 2\pi i$, resp. $\int_{q_{jn}} d\log(1 - x_j \cdots x_n) = 2\pi i$). Then

$$\begin{aligned} (\Theta(q_{j0}) - \mathrm{id})F_{1n}(\mathbf{x}) &= -\, 2\pi i \sum_{s=j}^{n-1} F_{1s}(\mathbf{x}) F_{s+1,n}(\mathbf{x}), \\ (\Theta(q_{1j}) - \mathrm{id})F_{1n}(\mathbf{x}) &= \ \ 2\pi i F_{1,j}(\mathbf{x}) F_{j+1,n}(\mathbf{x}), \\ (\Theta(q_{jn}) - \mathrm{id})F_{1n}(\mathbf{x}) &= -\, 2\pi i F_{1,j-1}(\mathbf{x}) F_{jn}(\mathbf{x}). \end{aligned}$$

Proof. The lemma follows from the monodromy property of $\mathfrak{L}_n(\mathbf{x})$. □

Combining Lemmas 3.9 to 3.11 we have

Theorem 3.12. *Let* $\mathcal{M}_{[n]}(\mathbf{x}) = \big[E_{\mathbf{i},\mathbf{j}}(\mathbf{x})\big]_{\mathbf{i},\mathbf{j}\in\mathfrak{S}_n}$ *where* $E_{\mathbf{i},\mathbf{j}}(\mathbf{x})$ *are defined by* (7)*. Let* $1 \le i \le j \le n$ *and* $q_{ij} \in \pi_1(S_n, \mathbf{x})$ *(resp.* $1 \le j < n$ *and* q_{j0}*) enclose* $\mathcal{D}_{ij} = \{x_i \dots x_j = 1\}$*, (resp.* $\mathcal{D}_{j0} = \{x_j = 0\}$*) only once but no other irreducible component of* X_n *such that* $\int_{q_{ij}} d\log(1 - x_i \dots x_j) = 2\pi i$ *(resp.* $\int_{q_{j0}} d\log x_j = 2\pi i$*). Then*

$$M(q_{j0}) = I + \big[n_{\mathbf{i},\mathbf{j}}\big]_{\mathbf{i},\mathbf{j}\in\mathfrak{S}_n}, \quad M(q_{ij}) = I + \big[m_{\mathbf{i},\mathbf{j}}\big]_{\mathbf{i},\mathbf{j}\in\mathfrak{S}_n},$$

where I *is the identity matrix of rank* 2^n*,*

$$n_{\mathbf{i},\mathbf{j}} = \begin{cases} -1 & \text{if } t_r \le j \le t_{r+1} - 2,\ r \ge 1,\ \mathbf{i} = \mathbf{j} + \mathbf{u}_{s+1} \text{ and } j \le s \le t_{r+1} - 2, \\ 0 & \text{otherwise,} \end{cases} \tag{15}$$

and

$$m_{\mathbf{i},\mathbf{j}} = \begin{cases} 1 & \text{if } t_r = i \le j \le t_{r+1} - 2,\ r \ge 1,\ \mathbf{i} = \mathbf{j} + \mathbf{u}_{j+1}, \\ -1 & \text{if } t_r + 1 \le i \le j = t_{r+1} - 1,\ r \ge 0,\ \mathbf{i} = \mathbf{j} + \mathbf{u}_i, \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Here $\mathbf{i}$ *and* $\mathbf{j}$ *satisfy the condition in Definition* 3.2(2) *in the cases of* $m_{\mathbf{i},\mathbf{j}} = \pm 1$ *and* $n_{\mathbf{i},\mathbf{j}} = -1$*.*

It follows immediately that we have

Corollary 3.13. *The monodromy representation of* $\mathcal{M}_{[n]}(\mathbf{x})$

$$\rho_{\mathbf{x}} : \pi_1(S_n, \mathbf{x}) \longrightarrow \mathrm{GL}_{2^n}(\mathbb{Z})$$

is unipotent.

3.3 Mixed Hodge structures of multiple logarithms

Having analyzed the monodromy properties of $\mathcal{M}_{[n]}(\mathbf{x})$ associated with the n-tuple logarithm we now turn to its mixed Hodge structures. Define a meromorphic connection ∇ on the trivial bundle

$$\mathbb{CP}^n \times \mathbb{C}^{2^n} \longrightarrow \mathbb{CP}^n \tag{17}$$

by

$$\nabla f = df - \boldsymbol{\omega} f,$$

where $f : S_n \to \mathbb{C}^{2^n}$ is a continuous section. This connection has regular singularities along the divisor $\mathcal{X}_n = X_n \cup (\mathbb{CP}^n \setminus \mathbb{C}^n)$ because $\boldsymbol{\omega}$ is integrable by (14) and all the 1-forms in $\boldsymbol{\omega}$ are logarithmic in any compactification of S_n. Proposition 3.7 further implies that the columns $(2\pi i)^{|\mathbf{j}|} C_{\mathbf{j}}(\mathbf{x})$ of $\mathcal{M}_{[n]}(\mathbf{x})$ satisfy $\nabla f = 0$ and are therefore flat sections of (17). Even though they are multi-valued, their $\mathbb{Z}$-linear span is well defined thanks to Theorem 3.12. Hence $V_{[n]}(\mathbf{x})$ forms a local system over S_n.

To define the MHS on $V_{[n]}$ one need provide two compatible filtrations: an increasing weight filtration $W_\bullet$ of $V_{[n]}(\mathbf{x})$ and a decreasing Hodge filtration $\mathcal{F}^\bullet$ of $V_{[n],\mathbb{C}} = V_{[n]}(\mathbf{x}) \otimes \mathbb{C}$. The weight filtration is defined by $W_{2k+1} = W_{2k}$ and

$$W_{-2k} V_{[n]}(\mathbf{x}) = \langle (2\pi i)^{|\mathbf{i}|} C_{\mathbf{i}} :\ |\mathbf{i}| \ge k \rangle_{\mathbb{Q}}.$$

In particular, $W_{-2k}V_{[n]}(\mathbf{x}) = 0$ if $k > n$ and $W_{-2k}V_{[n]}(\mathbf{x}) = V_{[n]}(\mathbf{x})$ if $k \le 0$. By regarding $e_{\mathbf{i}}$'s as column vectors one can define the Hodge filtration on $V_{[n],\mathbb{C}}$ by

$$\mathcal{F}^{-k}V_{[n],\mathbb{C}} := \langle e_{\mathbf{i}} : \ |\mathbf{i}| \le k \rangle_{\mathbb{C}}.$$

So in particular, $\mathcal{F}^{-k}V_{[n],\mathbb{C}} = 0$ for $k < 0$ and $\mathcal{F}^{-k}V_{[n],\mathbb{C}} = V_{[n],\mathbb{C}}$ for $k \ge n$.

By induction on n and using Lemma 3.4 it is easy to show that

$$\mathcal{F}^{-p} \cap W_{-2k}V_{[n],\mathbb{C}} = \begin{cases} 0 & \text{if } p \le k-1, \\ \langle (2\pi i)^{|\mathbf{i}|} e_{\mathbf{i}} : \ k \le |\mathbf{i}| \le p \rangle & \text{if } k \le p \le n, \\ \langle (2\pi i)^{|\mathbf{i}|} e_{\mathbf{i}} : \ k \le |\mathbf{i}| \le n \rangle & \text{if } p \ge n. \end{cases}$$

This implies that

$$\mathcal{F}^{-p} \operatorname{gr}^W_{-2k} V_{[n],\mathbb{C}} = \begin{cases} 0 & \text{if } p \le k-1, \\ W_{-2k}V_{[n],\mathbb{C}}/W_{-2k-1}V_{[n],\mathbb{C}} & \text{if } p \ge k. \end{cases}$$

In other words, $\mathcal{F}^q \operatorname{gr}^W_{-2k} V_{[n],\mathbb{C}} = 0$ for $q \ge -k+1$ and $\mathcal{F}^q \operatorname{gr}^W_{-2k} V_{[n],\mathbb{C}} = \operatorname{gr}^W_{-2k} V_{[n],\mathbb{C}}$ for $q \le -k$. This means that the Hodge filtration induces a pure HS of weight $-2k$ on each weight graded piece. Furthermore, it is not hard to see by checking the powers of $2\pi i$ appearing on the diagonal of $\mathcal{M}_{[n]}(\mathbf{x})$ that this induced structure on $\operatorname{gr}^W_{-2k} V_{[n],\mathbb{C}}$ is isomorphic to the direct sum of $\binom{n}{k}$ copies of the Tate structure $\mathbb{Z}(k)$ by Lemma 3.4.

4 Limit MHS of multiple logarithms

Let the monodromy of $\mathcal{M}_{[n]}(\mathbf{x})$ at any subvariety $\mathcal{D}$ of $\mathbb{CP}^n$ be given by the matrix $T_{\mathcal{D}}$ and the local monodromy logarithm by $N_{\mathcal{D}} = \log T_{\mathcal{D}}/2\pi i$. Note that $T_{\mathcal{D}}$ is unipotent so $N_{\mathcal{D}}$ is well-defined.

Now let us recall the construction of the unipotent variations of limit MHS at the "infinity" with *normal crossing*. Let S be a complex manifold of dimension d. Suppose that S is embedded in $\tilde{S}$, via the mapping j, such that $D = \tilde{S} - S$ is a divisor with normal crossings. Let $\mathbb{V}$ be any local system of complex vector spaces on S, and $\mathcal{V}$ the corresponding vector bundle. According to Deligne there is a canonical extension $\tilde{\mathcal{V}}$ of $\mathcal{V}$ over $\tilde{S}$ ([11, Proposition 5.2]). Moreover, when the local monodromy is nilpotent $\tilde{\mathcal{V}}$ is a subsheaf of $j_*\mathcal{V}$. The local picture of $S \subset \tilde{S}$ is $(\Delta^*)^r \times \Delta^{d-r} \subset \Delta^d$ where Δ is the unit disk and Δ^* is the punctured one. We let $t_1, \dots, t_r$ denote the variables on $(\Delta^*)^r$, and $N_1, \dots, N_r$ the (commuting) local nilpotent logarithms of the associated monodromy transformations of the fibre. For $z_1, \dots, z_r$ in the upper half-plane, the universal covering mapping for $(\Delta^*)^r$ is given by

$$t_j = \exp(2\pi i z_j), \quad j = 1, \cdots, r.$$

Let $v_1, \dots, v_m$ be a basis of the multi-valued sections of $\mathbb{V}$ over $(\Delta^*)^r \times \Delta^{d-r}$, the formula

$$[\tilde{v}_1, \dots, \tilde{v}_m] = [v_1, \dots, v_m] \exp\bigg(-\sum_{j=1}^r 2\pi i z_j N_j \bigg) = [v_1, \dots, v_m] \prod_{j=1}^r t_j^{-N_j}$$

determines a basis of the sections of $\mathcal{V}$ over Δ^d and these provide, by definition, the generators of $\tilde{\mathcal{V}}$ over Δ^d.

In our situation, although the divisor X_n is not normal crossing the image of the global holomorphic logarithmic forms in the complex of smooth forms on S_n is independent of the normal crossings compactification $\tilde{S}_n$ (cf. [24, Prop. (3.2)]). In fact, the forms we are considering lie in the subcomplex generated by 1-forms of the type df/f where f is a rational function. Such forms are automatically logarithmic in any compactification and therefore our connection is automatically regular. Hence the admissibility and the existence of the limit MHS is an automatic consequence of the admissibility of our variations restricted to every curve in S_n. Moreover, the pullback of our trivial bundle (17) restricted to S_n to $\tilde{S}_n$ is exactly Deligne's canonical extension of (17), and the pullbacks of the subbundles $\mathcal{F}^\bullet$ and $W_\bullet$ are the correct extended Hodge and weight subbundles. Therefore we have

Theorem 4.1. *The n-tuple logarithm underlies a good unipotent graded-polarizable variation of mixed Hodge-Tate structures $(V_{[n]}, W_\bullet, \mathcal{F}^\bullet)$ over*

$$S_n = \mathbb{C}^n \setminus \Big\{ \prod_{1\le j\le n} x_j(1-x_j) \prod_{1\le i<j\le n} (1 - x_i \dots x_j) = 0 \Big\}$$

with the weight-graded quotients gr^W_{-2k} being given by $\binom{n}{k}$ copies of the Tate structure $\mathbb{Z}(k)$.

Proof. It is clear that all the odd graded weight quotients are zero so that we can let the polarizations on the weight graded quotients gr^W_{-2k} be the ones that give each vector $2\pi i e_{\mathbf{j}}$ ($|\mathbf{j}| = k$) length 1. Then everything is clear except the Griffiths transversality condition. But this condition is also satisfied because $dC_{\mathbf{j}} = \boldsymbol{\omega} C_{\mathbf{j}}$ for every $\mathbf{j} \in \mathfrak{S}_n$ by Proposition 3.7. □

If we want to determine the limit MHS of multiple logarithms explicitly we can still apply the techniques used in the normal crossing case. We will carry this out only for the double logarithm case. The general picture is similar but much more complicated.

We have

$$\mathcal{M}_{1,1}(x,y) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathfrak{L}_1(y) & 1 & 0 & 0 \\ \mathfrak{L}_1(xy) & 0 & 1 & 0 \\ \mathfrak{L}_2(x,y) & \mathfrak{L}_1(x) & H(x,y) & 1 \end{bmatrix} \tau_{1,1}(2\pi i),$$

where $H(x,y) = \mathfrak{L}_1(y) - \mathfrak{L}_1(x) - \log x$ and $\tau_{1,1}(2\pi i) = \mathrm{diag}[1,\ 2\pi i,\ 2\pi i,\ (2\pi i)^2]$. To save space we let $M_{i,j}$ be the 4 by 4 matrix whose entries are all zero except that the (i,j)th entry is 1.

(i) Take the divisor $\mathcal{D}_{10} = \{x = 0\}$ and the tangent vector $\partial/\partial x$. We have

$$T_{\{x=0\}} = -M_{4,3}, \quad N_{\{x=0\}} = \frac{\log T_{\{x=0\}}}{2\pi i} = -\frac{1}{2\pi i} M_{4,3}.$$

Define $[s_0\ s_1\ s_2\ s_3] = \lim_{t\to 0}\mathcal{M}_{1,1}(t,y)\cdot\log t\cdot M_{4,3}/2\pi i$ then it's easy to see that

$$[s_0\ s_1\ s_2\ s_3] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathfrak{L}_1(y) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \mathfrak{L}_1(y) & 1 \end{bmatrix} \tau_{1,1}(2\pi i).$$

Let $V_{\mathbb{Q},\{x=0\}}$ be the $\mathbb{Q}$-linear span of s_0, s_1, s_2, s_3, and $V_{\mathbb{C},\{x=0\}} = \mathbb{C}\otimes V_{\mathbb{Q},\{x=0\}}$. Let $\{e_0, e_1, e_2, e_3\}$ be the standard basis of $\mathbb{C}^4$ where the only nonzero entry of e_j is at the $(j+1)$st component. Then the limit MHS on $\{(x,y) : x = 0, y \neq 1\}$ along $\partial/\partial x$ are given by

$$((V_{\mathbb{Q},\{x=0\}}, W_\bullet), (V_{\mathbb{C},\{x=0\}}, F^\bullet)),$$

where for $k = 0,\dots,3$

$$W_{-2k}V_{\mathbb{Q},\{x=0\}} = \langle s_k,\dots,s_3\rangle, W_{-2k} = W_{-2k+1} \tag{18}$$

and

$$F^{-k}V_{\mathbb{C},\{x=0\}} = \langle e_0,\dots,e_k\rangle. \tag{19}$$

(ii) A similar calculation shows that on the divisor $\mathcal{D}_{11} = \{(1,y) : y \neq 1\}$ and along the tangent vector $\partial/\partial x$ if one sets $[s_0\ s_1\ s_2\ s_3] = \lim_{t\to 1}\mathcal{M}_{1,1}(t,y)\cdot\log(1-t)(M_{4,2} - M_{4,3})/2\pi i$ then one has the limit MHS

$$[s_0\ s_1\ s_2\ s_3] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \mathfrak{L}_1(y) & 1 & 0 & 0 \\ \mathfrak{L}_1(y) & 0 & 1 & 0 \\ \mathfrak{L}_2(1,y) & 0 & \mathfrak{L}_1(y) & 1 \end{bmatrix} \tau_{1,1}(2\pi i).$$

It is easy to see by differentiation that $\mathfrak{L}_2(1,y) = \big(\mathfrak{L}_1(y)\big)^2/2$.

We can similarly deal with the next two cases:

(iii) On $\mathcal{D}_{22} = \{(x,1) : x \neq 0,1\}$ along the tangent vector $\partial/\partial y$.

(iv) On $\mathcal{D}_{12} = \{(1/y,y) : y \neq 0,1\}$ along the tangent vector $\partial/\partial x$. (The limit is given by $(x,y) \longrightarrow (1/y,y)$ for every fixed $y \neq 0,1$.)

For 0-dimensional singularities we only have the following two cases:

(v) $\mathcal{D}_{10}\cap\mathcal{D}_{22} = (0,1)$. From (i) we see that there are limit MHS on the open set $\mathcal{D}_{10}\setminus\{(0,1)\}$ of $\mathcal{D}_{10}$. We now can easily extend these MHS to $(0,1)$ along the vector $\partial/\partial y$ and find the limit MHS to be the $\mathbb{Q}$-linear span of $s_0,\cdots,s_3$ where

$$[s_0\ s_1\ s_2\ s_3] = \tau_{1,1}(2\pi i).$$

If we start with (iii) and then extend the MHS to $(0,1)$ along tangent vector $\partial/\partial x$ we will get the same limit MHS.

(vi) $\mathcal{D}_{11}\cap\mathcal{D}_{12} = \mathcal{D}_{12}\cap\mathcal{D}_{22} = \mathcal{D}_{11}\cap\mathcal{D}_{22} = (1,1)$. We can start with either case (ii) or (iii) or (iv). Extending the limit MHS of case (ii) we see immediately that the along the tangent vector $\partial/\partial y$ the limit MHS at $(1,1)$ is given by the $\mathbb{Q}$-linear span of

$$[s_0\ s_1\ s_2\ s_3] = \tau_{1,1}(2\pi i). \tag{20}$$

If we start with case (iii) and then use tangent vector $\partial/\partial x$ we find that only the lower left corner entry is different from the above. Instead of 0 it is

$$E_{4,1} = \lim_{x\to 1} Li_2(\frac{x}{x-1}) + \frac{1}{2}\log^2(1-x) - \log x \log(1-x) = -Li_2(1) = -\frac{\pi^2}{12},$$

since $Li_2(1-t)+Li_2(1-1/t)+\log^2 t/2 = 0$ for any $t \neq 0$. But if we take $s'_0 = s_0 - s_3/48$ we get the same basis as in (20). The same phenomenon occurs if we start with case (iv) and then use tangent vector $\partial/\partial y$.

Problem 4.2. We find that in higher weight cases the limit MHS of the multiple logarithm sometimes correspond to MHS of some multiple *poly*logarithm of the same weight. Can all the multiple *poly*logarithm variations of MHS be produced this way by multiple logarithms?

5 Some general results and problems

One can similarly generalize the above theory to multiple *poly*logarithms. In single variable case, Deligne defined a variation of mixed Hodge-Tate structures related to the classical n-logarithm (cf. [23]). In the case of two variables, we have the next result for weight 3.

Theorem 5.1. *Each of the weight three depth two multiple polylogarithms underlies a good variation of mixed Hodge-Tate structures over* $S_2 = \mathbb{C}^2 \setminus \{xy(1-x)(1-y)(1-xy)=0\}$. *For* $Li_{2,1}$ *the graded weight quotients are isomorphic to* $\mathbb{Z}(0)$, $\mathbb{Z}(1)\oplus\mathbb{Z}(1)$, $\mathbb{Z}(2)\oplus\mathbb{Z}(2)$, *and* $\mathbb{Z}(3)$, *respectively. For* $Li_{1,2}$ *they are isomorphic to* $\mathbb{Z}(0)$, $\mathbb{Z}(1)\oplus\mathbb{Z}(1)$, $\mathbb{Z}(2)\oplus\mathbb{Z}(2)\oplus\mathbb{Z}(2)$, *and* $\mathbb{Z}(3)$, *respectively.*

By this theorem and previous results on Li_3 and $Li_{1,1,1}$ we have now completely settled the cases of weight 3 multiple polylogarithms. We now can generalize Theorem 5.1 to

Theorem 5.2. *The double polylogarithm* $Li_{r,s}$ *underlies a good unipotent graded-polarizable variation of a mixed Hodge-Tate structure with the graded weight piece* gr^W_{-2k} *being direct sums of* c_k *copies of* $\mathbb{Z}(k)$ *where*

$$c_k = \begin{cases} d_k(r,s)+1 & \text{if } r \neq k = s, \\ d_k(r,s) & \text{otherwise,} \end{cases}$$

and

$$d_k(r,s) = \begin{cases} 0 & \text{if } k<0 \text{ or } k>r+s, \\ k+1 & \text{if } 0 \le k \le \min\{r,s\}, \\ \min\{r,s\}+1 & \text{if } \min\{r,s\} \le k \le \max\{r,s\}, \\ r+s+1-k & \text{if } \max\{r,s\} \le k \le r+s. \end{cases}$$

Among all the double polylogarithms $Li_{r,r}(x,y)$ is the most regular. It satisfies $c_0 = c_{2r} = 1, c_1 = c_{2r-1} = 2, \dots, c_{r-1} = c_{r+1} = r, c_r = r+1$.

In general, we expect the multiple polylogarithm $Li_{m_1,\dots,m_n}(\mathbf{x})$ underlies a good variation of mixed Hodge-Tate structures over S_n with the graded weight piece gr^W_{-2k} being direct sums of c_k copies of $\mathbb{Z}(k)$ for some positive integer c_k. It would be very interesting to solve

Problem 5.3. (1) Find a closed formula of c_k only depending on $m_1, \dots, m_n$ and k.

(2) Determine the variation matrix $\mathcal{M}_{m_1,\dots,m_n}(\mathbf{x})$ explicitly.

(3) Determine the connection matrix $\boldsymbol{\omega}$ explicitly.

(4) Determine the monodromy actions explicitly.

6 Single-valued version of multiple polylogarithms

If part (2) of Problem 5.3 is solved then following an idea of Beilinson and Deligne [1] as given in [4] one can easily discover the single-valued variant of $Li_{m_1,\dots,m_n}(x_1,\dots,x_n)$ which we denote by $\mathcal{L}_{m_1,\dots,m_n}(x_1,\dots,x_n)$ which should be a real analytic function. In what follows we outline the procedure for multiple logarithms only.

6.1 General procedures

For any $n \geq 2$ let $L_{[n]} = L_{[n]}(\mathbf{x}) = [C_{\mathbf{0}} \ \dots \ C_{\mathbf{1}}]$ be the matrix with 2^n columns $C_{\mathbf{j}}$ ($\mathbf{j} \in \mathcal{S}_n$) as before and $\mathcal{M}_{[n]} = \mathcal{M}_{[n]}(\mathbf{x}) = L_{[n]}(\mathbf{x})\tau_{[n]}(2\pi i)$ where

$$\tau_{[n]}(\lambda) = \mathrm{diag}\big[\lambda^{|\mathbf{j}|}\big]_{\mathbf{j}\in\mathcal{S}_n}.$$

Define the matrix

$$B_{[n]} = B_{[n]}(\mathbf{x}) = \tau_{[n]}(i)\mathcal{M}_{[n]}\overline{\mathcal{M}}_{[n]}^{-1}\tau_{[n]}(i),$$

where $\overline{\mathcal{M}}_{[n]}$ is the complex conjugation of $\mathcal{M}_{[n]}$. From our calculation of the monodromy we see that B is a single-valued matrix function defined over S_n. Moreover

$$\overline{B}_{[n]} = B_{[n]}^{-1}$$

since $\overline{\tau_{[n]}(i)} = \tau_{[n]}(i)^{-1}$. Now that $B_{[n]} = I + N$ with I the identity matrix and N a nilpotent matrix we see that $\log B$ is well defined and satisfies

$$\overline{\log B_{[n]}} = -\log B_{[n]},$$

namely, $\log B_{[n]}$ is a pure imaginary matrix. Then we define $-1/(2i)$ times the lower left corner entry of $\log B$ to be $\mathcal{L}_{[n]}(\mathbf{x})$ which is a single-valued real analytic variant of the multiple logarithm $\mathfrak{L}_n(\mathbf{x})$.

Remark 6.1. Our method is slightly different from that in [1]. In fact when we are in the polylogarithm case the matrix B constructed as above is the conjugate of the one in [1] by $\tau(i)$.

6.2 Single-valued multiple polylogarithms of lower weights

By the above procedure we find the single-valued double logarithm

$$\mathcal{L}_{1,1}(x,y) = \mathcal{L}_2\Big(\frac{xy-y}{1-y}\Big) - \mathcal{L}_2\Big(\frac{y}{y-1}\Big) - \mathcal{L}_2(xy) \tag{21}$$

where the dilogarithm function

$$\mathcal{L}_2(z) = \mathrm{Im}\,\big(Li_2(z)\big) + \arg(1-z)\log|z|.$$

The function $\mathcal{L}_{1,1}(x,y)$ satisfies the functional equations

$$\mathcal{L}_{1,1}(x,y) = -\mathcal{L}_{1,1}\Big(1-x, \frac{y}{y-1}\Big)$$

by the functional equations $\mathcal{L}_2(x) = -\mathcal{L}_2(1-x) = -\mathcal{L}_2(1/x)$.

Similarly, we compute the single-valued versions of $Li_{2,1}(x,y)$ and $Li_{1,2}(x,y)$ as follows:

$$\begin{aligned}\mathcal{L}_{1,2}(x,y) &= \mathrm{Re}\,Li_{1,2}(x,y) - \arg(1-xy)\big[\mathcal{L}_2(x)+\mathcal{L}_2(y)\big] + \log|1-x|\,\mathrm{Re}\,Li_2(y)\\ &- \log|y|\,\mathrm{Re}\,Li_{1,1}(x,y) - \log|1-x^{-1}|\,\mathrm{Re}\,Li_2(xy) - \frac{1}{3}\log|xy^2|\log|1-xy|\log\Big|1-x^{-1}\Big|\\ &+ \frac{1}{3}\log|y|\big(2\log|1-y|\log|1-x| + \log|1-xy|\log|x(1-y)|\big),\end{aligned}$$

and

$$\mathcal{L}_{1,2}(x,y) = -\mathcal{L}_{2,1}(y,x) - \mathcal{L}_3(xy), \tag{22}$$

where $\mathcal{L}_3$ is the single-valued trilogarithm given by (cf. [37])

$$\mathcal{L}_3(z) = \mathrm{Re}\,\big(Li_3(z)\big) - \log|z|\,\mathrm{Re}\,\big(Li_2(z)\big) - \frac{1}{3}(\log|z|)^2\log|1-z|.$$

One should compare (22) with the identity

$$Li_{1,2}(x,y) + Li_{2,1}(y,x) + Li_3(xy) = -\log(1-x)Li_2(y).$$

Finally we find the interesting identity

$$\begin{aligned}\mathcal{L}_{1,1,1}(x,y,z) =& \mathcal{L}_3\Big(\frac{(y-1)(1-xyz)}{y(1-x)(1-z)}\Big) + \mathcal{L}_3\Big(\frac{y}{y-1}\Big) + \mathcal{L}_3(xy) - \mathcal{L}_3\Big(\frac{1-xyz}{1-x}\Big)\\ &- \mathcal{L}_3\Big(\frac{1-xyz}{xy(1-z)}\Big) - \mathcal{L}_3\Big(\frac{y-yz}{y-1}\Big) - \mathcal{L}_3\Big(\frac{y-xy}{y-1}\Big) + \mathcal{L}_3(1-x).\end{aligned}$$

We remind the readers that such identities in higher weight cases do not exist in general. For example, $\mathcal{L}_{2,2}(x,y)$ cannot be expressed by $\mathcal{L}_4$ only (see the explanation on [15, pp.244–245]).

6.3 A problem of multiple Dedekind zeta values

In general there exists single-valued real analytic version of the multiple polylogarithm $Li_{m_1,\ldots,m_n}(\mathbf{x})$ which we denote by $\mathcal{L}_{m_1,\ldots,m_n}(\mathbf{x})$. For $m_n \geq 2$ the value of this function is given by the power series expansion (1) when $|x_i| \leq 1$. We end our exposition by posing a problem generalizing Zagier conjecture about special values of Dedekind zeta function over number fields.

Denote by $\mathcal{O}_F$ the ring of integers of a number field F and I_F the set of integral ideals of $\mathcal{O}_F$. Let $\mathcal{N}$ be the norm from F to $\mathbb{Q}$. Then we define the multiple Dedekind zeta function of depth d over F as

$$\zeta_F(s_1,\ldots,s_d) = \sum_{\substack{\mathfrak{n}_1,\ldots,\mathfrak{n}_d \in I_F \\ \mathcal{N}(\mathfrak{n}_1) < \cdots < \mathcal{N}(\mathfrak{n}_d)}} \mathcal{N}(\mathfrak{n}_1)^{-s_1} \cdots \mathcal{N}(\mathfrak{n}_d)^{-s_d}.$$

This function is well defined for $\mathrm{Re}(s_1) > 0, \ldots, \mathrm{Re}(s_{d-1}) > 0, \mathrm{Re}(s_d) > 1$.

Problem 6.2. For any integers $m_1, \ldots, m_{d-1} \geq 1$ and $m_d \geq 2$, is there an expression of $\zeta_F(m_1,\ldots,m_d)$ in terms of a determinant of $\mathcal{L}_{m_1,\ldots,m_d}$ evaluated at F rational points up to some factors determined only by the number field F (such as the discriminant, the number of real and complex embeddings, etc.)?

When $F = \mathbb{Q}$ the problem has an easy answer:

$$\zeta_{\mathbb{Q}}(m_1,\ldots,m_d) = \mathcal{L}_{m_1,\ldots,m_d}(1,\ldots,1).$$

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