Iwasawa Theory of $\mathbb{Z}^d_p$-Extensions over Global Function Fields

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Abstract. In this paper we study the Iwasawa theory of $\mathbb{Z}^d_p$-extensions of global function fields $k$ over finite fields of characteristic $p$. When $d = 1$ we first show that Iwasawa invariants are well defined under the assumption that only finitely many primes are ramified in the extension, then we prove that the Iwasawa $\mu$-invariant can be arbitrarily large for some extension of any given base field $k$. After giving some general results of $\mathbb{Z}^d_p$-extensions we finally study the behaviors of Iwasawa invariants in $\mathbb{Z}^2_p$-extensions in some detail, especially when the extensions contain a constant $\mathbb{Z}_p$-extension. The whole paper is expository in nature though it contains some new results.

1 Introduction

Around 1940 A. Weil developed a theory which relates the zeta function of the function fields to the characteristic polynomials of Frobenius acting on $p$-power order torsion in the Jacobian of the corresponding curve. He chose $p$ to be any prime different from the characteristic of the field. In looking for an analogue of this theory for number fields, beginning in late 1950’s, K. Iwasawa carried out the fundamental work of what is now called the Iwasawa theory in a series of papers. To get a better analogy between $p$-power order torsion in the Jacobian on the function field side and the $p$-Sylow subgroup of the class group on the number field side, Iwasawa considered field tower obtained by adjoining the $p$-power roots of unity or more generally $\mathbb{Z}_p$-extensions for some fixed $p$. By definition, a $\mathbb{Z}_p$-extension of a field $k$ is an infinite Galois extension $k_{\infty}/k$ with the Galois group isomorphic to the additive group of $p$-adic integers $\mathbb{Z}_p$. It is possible to regard a $\mathbb{Z}_p$-extension as a sequence of fields

$$k = k_0 \subset k_1 \subset \cdots \subset k_{\infty} = \bigcup_{n=1}^{\infty} k_n$$

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with $\text{Gal}(k_n/k) \cong \mathbb{Z}/p^n\mathbb{Z}$. We say $k_n$ is the $n$-th layer of the $\mathbb{Z}_p$-extension. Iwasawa at first named them $\Gamma$-extensions and later on he and others switched to the now standardized name "$\mathbb{Z}_p$-extensions". One of the prevailing objectives of Iwasawa theory is to study the behavior of the ideal class group of $k_n$ as $n$ varies.

Classically, there are several approaches to this problem: (A) via class number formula and $p$-adic $L$-functions; (B) via class number formula and $p$-adic measures on $\mathbb{Z}_p$; (C) via the theory of noetherian modules over Iwasawa algebra $\lim_n \mathbb{Z}_p[\mathbb{Z}/p^n\mathbb{Z}]$. Approach (A) works particularly well for the so called cyclotomic $\mathbb{Z}_p$-extensions (i.e. the compositum of all the cyclotomic number fields with the base field). But it apparently fails for more general fields which do not possess a good class number formula. (B) was found by Sinnott (see [26] and [27]) when he studied the work of Ferrero and Washington. Neither (A) nor (B) can be used to handle the function field case because good analogues of $p$-adic $L$-functions and $p$-adic measures are still unborn for function fields. Approach (C) was the one initiated by Iwasawa and further developed by Serre, Greenberg, Cuoco, Monsky and many others. This method can be combined with class field theory and Kummer theory to get a lot of nice results.

The last approach of the above also has the merit of being generalizable to function field case, which is what we shall use in the present paper. More precisely we shall return to the function fields $k$ of transcendental degree one over a finite field $\mathbb{F}_q$ and deal with the situation when $p$ is the characteristic of $k$. Our primary interest lies in the ideal class numbers of different layers in any $\mathbb{Z}_p$-extension over $k$. We are also concerned with $\mathbb{Z}_p^d$-extension $K_\infty/k$ such that the Galois group $\Gamma$ is isomorphic to $d$ copies of $\mathbb{Z}_p$. Let $K_n$ be the fixed field of $\Gamma_\infty$. Then it is a standard fact that the group of divisor classes of $K_n$ of degree zero $C_0(K_n)$ is finite. We will denote the $p$-part subgroup of $C_0(K_n)$ and its order by $X_n$ and $h_n = p^{e_n}$ respectively. There is a rather complete Iwasawa theory in the number field case for $d = 1$. In this case, Iwasawa showed that there are constants $\mu$, $\lambda$ and $\nu$ such that $e_n = \lambda n + \mu p^n + \nu$. Cuoco made the first generalization of Iwasawa’s formula to $d = 2$ in his Ph.D thesis (see [2]). Exploiting Greenberg’s result (see [8]) about $\Lambda_d$-modules and Monsky’s result (see [18]) about $d$-variable power series over $\mathbb{Z}_p$, Cuoco and Monsky [4] were able to extend Iwasawa’s formula to all positive integers $d$.

Given a global function field $k$, an obvious $\mathbb{Z}_p$-extension is the constant field $\mathbb{Z}_p$-extension. If $k_\infty/k$ is such an extension, then there are no ramified
primes in \( k_\infty \). In contrast to the constant extension, by a *geometric extension* we mean an extension which does not contain any constant field extension. A geometric \( \mathbb{Z}_p \)-extension of \( k \) is the one such that \( k_n/k \) is geometric for every \( n \)-th layer \( k_n \). It is easy to see that any non-constant \( \mathbb{Z}_p \)-extension of \( k \) becomes a geometric \( \mathbb{Z}_p \)-extension by throwing away first finitely many layers. Thus a natural question is: how many geometric \( \mathbb{Z}_p \)-extensions does \( k \) have? The answer is a little surprising compared with the number field case: there are infinitely many. This can be derived from class field theory without too much difficulty (see [7, Theorem 3]).

Another difference between number field and function field \( \mathbb{Z}_p \)-extensions is that for a given \( \mathbb{Z}_p \)-extension of number field, there are only finitely many ramified primes in this extension. But in the function field case, there might be infinitely many primes of \( k \) which are ramified in a \( \mathbb{Z}_p \)-extension over \( k \). This is not too hard to prove by using the ideas from [7]. Throughout this paper, we will assume that all \( \mathbb{Z}_d \)-extensions \( K_\infty/k \) satisfies the following assumption:

**FRP** Only finitely many primes of \( k \) ramify in \( K_\infty \).

If the assumption fails when \( d = 1 \) Gold and Kisilevsky ([7, Theorem 2]) showed essentially that one cannot define the Iwasawa invariants properly for the \( \mathbb{Z}_p \)-extension. In addition to this difficulty, while adapting the proofs of some key results in number fields to the function field \( \mathbb{Z}_d \)-extensions, we find that this assumption is actually indispensable (for instance, the proof of Theorem 4.1 and 5.1 would fail if the number of ramified primes of \( k \) is infinity.)

We organize this paper as follows. In §2, we give a brief argument about how \( h_n \) changes when \( n \) increases and define the Iwasawa \( \mu \)-invariant and \( \lambda \)-invariant under the assumption **FRP**. Since the proofs are quite similar to the classical case we omit most of them and refer the reader to the very well written book by L. Washington [29] and other authors' papers on this topic.

In §3, we show that \( \mu \)-invariant can be arbitrarily large for geometric \( \mathbb{Z}_p \)-extensions even under the assumption **FRP**. Similar result in number field case holds as showed by Iwasawa (see [13] or [14, p. 310]). Here we are unable to provide a good analogue of the following wonderful result in number field case: if \( k \) is abelian over \( \mathbb{Q} \) then the \( \mu \)-invariant of any cyclotomic \( \mathbb{Z}_p \)-extensions over \( k \) vanishes ([6] or [26]). Inspite of very good analogues of cyclotomic extensions ([10]) the Galois groups are too large to handle at present (see [9] and [16]).
In §4, we present some general results of \( \mathbb{Z}_p \)-extensions over \( k \). As an application, in the last section we will discuss the growth rate of Iwasawa invariants when the base field \( k \) changes in another \( \mathbb{Z}_p \)-extension. Especially, if \( k_\infty/k \) is a geometric \( \mathbb{Z}_p \)-extension with constant field \( \mathbb{F}_q \) and \( F_n \) is an extension of \( \mathbb{F}_q \) of degree \( p^n \), then \( k_\infty F_n/kF_n \) is also a geometric \( \mathbb{Z}_p \)-extension. Let \( \lambda_n \) and \( \mu_n \) be the corresponding Iwasawa invariants. We will show that (Theorem 5.7) there exists some constant \( c \) such that for \( n \) sufficiently large \( \lambda_n = \lambda(k_\infty/k) \) and \( \mu_n = \mu_{n-1} + c \).

2 Iwasawa invariants in \( \mathbb{Z}_p \)-extensions

Let \( k_\infty \) be a geometric \( \mathbb{Z}_p \)-extension of \( k \) such that \( k = k_0 \subset k_1 \subset \ldots \subset k_n \subset \ldots \subset k_\infty \), where \( k_n \) is the \( n \)-th layer of \( k_\infty/k \). In this section we will define the Iwasawa invariants under the assumption FRP in the introduction. Put \( \Gamma = \text{Gal}(k_\infty/k) = \langle \gamma_0 \rangle \). Let \( L \) be the maximal unramified abelian \( p \)-extension of \( k_\infty \). Let \( X \) and \( G \) be the Galois group of \( L/k_\infty \) and \( L/k \) respectively. Then one has the following diagram

\[
\begin{array}{ccc}
k_\infty & \xrightarrow{X} & L \\
\downarrow & & \downarrow \\
k & \xrightarrow{G} & \Gamma \end{array}
\]

One sees that \( \Gamma \cong G/X \) and \( \Gamma \) acts on \( X \) by extending to \( G \) and makes \( X \) into a \( \Gamma \)-module. More precisely, let

\[
0 \rightarrow X \rightarrow G \xrightarrow{\pi} \Gamma \rightarrow 0
\]

be a short exact sequence. For any \( x \in X \) and any \( r \in \Gamma \), choose any \( g \in G \) such that \( \pi(g) = r \), then define \( x^r = gxg^{-1} \). It is easily to see that this is well-defined. In fact, let \( g_1 \) be another element of \( G \) such that \( \pi(g_1) = r \), then \( g^{-1}g_1 = x_0 \) for some \( x_0 \in X \). Notice that \( X \) is abelian, so

\[
x^r = gxg^{-1} = gx_0^{-1}xx_0g^{-1} = g_1xg^{-1}
\]

Let \( \Lambda \) be the completed group ring \( \mathbb{Z}_p[[\Gamma]] \) which is isomorphic to \( \mathbb{Z}_p[[T]] \) under the homomorphism sending \( \gamma_0 - 1 \) to \( T \). As a projective limit of finite \( p \)-groups \( X \) is a compact \( \Gamma \)-module, which makes \( X \) into a compact \( \Lambda \)-module. We will show that \( X \) is noetherian \( \Lambda \)-torsion module and retrieve some information for \( X_n \) from some submodules of \( X \).
**Lemma 2.1** Let \( L \) be the maximal unramified abelian \( p \)-extension of \( k_\infty \), \( K_0 \) be the maximal unramified abelian \( p \)-extension of \( k \) contained in \( L \). Let \( k_\infty \) be a geometric \( \mathbb{Z}_p \)-extension of \( k \). Then there must be some prime of \( k \) which is ramified in \( k_\infty/k \).

**Proof.** Suppose that \( k_\infty \) is a geometric \( \mathbb{Z}_p \)-extension of \( k \) such that \( k_\infty \) is unramified over \( k \). Let \( \tilde{\mathbb{F}}_q \) be the constant \( \mathbb{Z}_p \)-extension of \( \mathbb{F}_q \). Then clearly \( k_\infty \tilde{\mathbb{F}}_q \) is unramified over \( k \) with Galois group \( \mathbb{Z}_p \times \mathbb{Z}_p \). By the class field theory,

\[
\text{Gal}(K_0/k) \cong \mathbb{Z}_p \times C_0(k).
\]

One finds a contradiction by noticing that \( C_0(k) \) is finite while \( k_\infty \tilde{\mathbb{F}}_q \subseteq K_0 \) by Lemma 2.2.

For a geometric \( \mathbb{Z}_p \)-extension, there are possibly infinitely many primes of \( k \) which are ramified in this extension. Now we assume that \( k_\infty/k \) satisfies the hypothesis FRP in the introduction. The inertia groups of the ramified primes are subgroups of \( \mathbb{Z}_p \) which are of the form \( p^d \mathbb{Z}_p \) for some \( d \). Thus, we can choose \( n_0 \) such that all the ramified primes are totally ramified in \( k_\infty/k_{n_0} \). Hence, without loss of generality we can assume that ramified primes are totally ramified in \( k_\infty/k \).

**Lemma 2.2** Notation as in Lemma 2.1. Then \( K_0 \) is the maximal unramified abelian extension of \( k_\infty \tilde{\mathbb{F}}_q \) contained in \( L \).

**Proof.** Let \( K_0' \) be the maximal unramified \( p \)-abelian extension of \( k_\infty \tilde{\mathbb{F}}_q \) such that \( K_0 \subseteq K_0' \subseteq L \). Then one has the following graph of field extensions

\[
\begin{array}{c}
\text{k_\infty} \\
\text{k} \\
\text{k_\infty} \tilde{\mathbb{F}}_q \\
\text{k} \tilde{\mathbb{F}}_q \\
\text{K_0'} \\
\end{array}
\]

For any \( \sigma \in \text{Gal}(L/k) \), \( \sigma(K_0') \) is the maximal unramified \( p \)-extension of \( \sigma(k_\infty \tilde{\mathbb{F}}_q) = k_\infty \tilde{\mathbb{F}}_q \). Since the maximal unramified extension is unique, one has \( \sigma(K_0') = K_0' \), whence \( K_0'/k \) is Galois. Now \( k_\infty/k \) is totally ramified and \( K_0'/k \) is unramified, so one has \( \text{Gal}(K_0'/k) \cong \text{Gal}(K_0'/k_\infty/k_\infty) \), which is abelian by the definition of \( L \) and \( K_0' \). Thus \( K_0 = K_0' \) as desired.

Now we know \( X \cong \text{Gal}(L/k_\infty \tilde{\mathbb{F}}_q) \times \mathbb{Z}_p \) as abelian groups (see [7, Proposition 2]), where \( \tilde{\mathbb{F}}_q \) is the constant \( \mathbb{Z}_p \)-extension of \( \mathbb{F}_q \) and \( \mathbb{Z}_p \cong \text{Gal}(k_\infty \tilde{\mathbb{F}}_q/k_\infty) \).
It is easy to see that \( k_{\infty} \bar{\mathbb{F}}_q \) is a Galois extension over \( k \) since both \( k_{\infty}/k \) and \( k \bar{\mathbb{F}}_q/k \) are Galois. Thus \( \text{Gal}(L/k_{\infty} \bar{\mathbb{F}}_q) \) is a normal subgroup of \( G \) and so is a submodule of \( X \) as \( \Lambda \)-module.

**Lemma 2.3** Let \( \Gamma_1 \) be the Galois group of \( k_{\infty} \bar{\mathbb{F}}_q/k \bar{\mathbb{F}}_q \) and \( Y = \text{Gal}(L/k_{\infty} \bar{\mathbb{F}}_q) \) as a \( \Lambda \)-module. Then \( Y \) as a \( \Gamma_1 \)-module is isomorphic to \( Y \) as a \( \Gamma \)-module.

**Proof.** Let \( f : \sigma \rightarrow \sigma|_{k_{\infty}} \) be the restriction of \( \Gamma \) to \( \Gamma_1 \). Define \( G_1 = \text{Gal}(L/k \bar{\mathbb{F}}_q) \). Then one has the following diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & G_1 & \overset{\pi_1}{\longrightarrow} & \Gamma_1 & \longrightarrow & 0 \\
& & \downarrow{i} & & \downarrow{i} & & \downarrow{f} & & \\
0 & \longrightarrow & X & \longrightarrow & G & \overset{\pi}{\longrightarrow} & \Gamma & \longrightarrow & 0.
\end{array}
\]

For any \( y \in Y \) and \( r \in \Gamma \), let \( g \) be an element of \( G \) such that \( \pi(g) = r \), then \( y^r = grg^{-1} \). Now let \( r' = f^{-1}(r) \) and \( g_1 \in G_1 \) such that \( \pi_1(g_1) = r' \). Then \( \pi(g) = f(\pi_1(g_1)) = \pi(i(g_1)) = r \), hence \( g^{-1}g_1 \) is an element of \( X \). But \( X \) is abelian, so

\[
y^r = g_1yg_1^{-1} = g_1(g^{-1}g_1)^{-1}y^{-1}g_1g_1^{-1} = yy^{-1} = y^r
\]

as desired. \[\square\]

From this lemma, it suffices to study \( Y \) as a \( \Gamma_1 \)-module. Let \( P_1, \ldots, P_s \) be the primes which are ramified in \( k_{\infty} \bar{\mathbb{F}}_q/k \bar{\mathbb{F}}_q \). Let \( I_i \subseteq G_1 \) be the inertia group, then one has \( I_i \cap Y = 1 \) since \( L/k_{\infty} \bar{\mathbb{F}}_q \) is unramified. Now that \( k_{\infty} \bar{\mathbb{F}}_q/k \bar{\mathbb{F}}_q \) is totally ramified at \( P_i \),

\[
I_i \hookrightarrow G_1/Y = \Gamma_1
\]

is bijective since \( L/k_{\infty} \bar{\mathbb{F}}_q \) is maximal unramified. So

\[
G_1 = I_i Y = Y I_i, \quad \text{for all } i = 1, \ldots, s.
\]

Let \( \sigma_i \in I_i \) map to the generator \( \gamma_0 \) of \( \Gamma_1 \). Then \( \sigma_i \) must be a topological generator of \( I_i \) and \( \sigma_i = a_i \sigma_1 \) for some \( a_i \in Y \) since \( I_i \subseteq Y I_1 \). Notice that \( a_1 = 1 \).

**Lemma 2.4** Let \( G'_1 \) be the closure of the commutator subgroup of \( G_1 \). Then

\[
G'_1 = Y^{70^{-1}} = TY
\]

.
**Proof.** See [29, Lemma 13.14, p. 278]. □

**Lemma 2.5** Let $Y_0$ be the submodule of $Y$ generated by $a_i, i = 2, \ldots, s$ and by $Y^{-\gamma_0} = TY$. Let $X_n$ be the $p$-part of $C_0(k_n)$ and $Y_n = v_n Y_0$, where

$$v_n = 1 + \gamma_0 + \gamma_0^2 + \ldots + \gamma_0^{p^n - 1} = \frac{(1 + T)^{p^n} - 1}{T}.$$  

Then for all $n \geq 0$

$$X_n \cong Y/Y_n.$$  

**Proof.** See [29, Lemma 13.15, p. 278]. □

Now we state some facts about $\Lambda$-modules.

**IW1.** ([29, Lemma 13.16, p. 279]) Let $M$ be a compact $\Lambda$-module. Then $M$ is finitely generated over $\Lambda$ if and only if $M/(p, T)M$ is finite.

**IW2.** ([29, p. 271]) Two $\Lambda$-modules $M$ and $M'$ are said to be pseudo-isomorphic, denoted by $M \sim M'$, if there is a homomorphism $M \rightarrow M'$ with finite kernel and cokernel.

**IW3.** ([29, Theorem 13.12, p. 271]) If $M$ is a finitely generated $\Lambda$-module. Then

$$M \sim \Lambda^r \oplus \bigoplus_{i=1}^s \Lambda/(p^{m_i}) \oplus \bigoplus_{j=1}^l \Lambda/(f_j(T)^{l_j}),$$

where $r, s, m_i, l_j \in \mathbb{Z}$ and $f_j$ are distinguished polynomials. Conventionally $\mu(M) = \sum_{i=1}^s m_i$ is called the Iwasawa $\mu$-invariant and $\lambda(M) = \sum_{j=1}^l l_j \deg(f_j)$ is called the Iwasawa $\lambda$-invariant. Moreover we can make the $f_j$’s irreducible.

**IW4.** ([29, Proposition 13.21, Proposition 13.19, p. 283]) Let

$$v_{n,e} = 1 + \gamma_0^p + \gamma_0^{p^e} + \ldots + \gamma_0^{p^n - p^e} = v_n/v_e,$$ for some $0 < e < n$.

If $M$ is noetherian and torsion then there exists some constant $\nu$ such that

$$|M/v_{n,e}M| = p^{\mu p^n + \lambda n + \nu}, \quad \text{for all } n \gg 0.$$
Suppose $Y$ and $E$ are $\Lambda$-modules such that $Y \sim E$ and $Y/v_{n,e}Y$ is finite for all $n \geq e$. Then, there exists some constant $\nu$ such that

$$|Y/v_{n,e}Y| = p^\nu |E/v_{n,e}E|$$

for all $n \gg 0$.

**Theorem 2.6** Both $\Lambda$-modules $X = \text{Gal}(L/k_\infty)$ and $Y = \text{Gal}(L/k_\infty \tilde{F}_q)$ are noetherian torsion $\Lambda$-module.

**Proof.** In Lemma 2.5 we have seen that $X_n \cong Y/Y_n$, where $X_n$ is the $p$-part of the class group of $k_n$, which is finite for all $n$. Since $v_1 = T \in (p, T)$, one sees that $Y_0/(p, T)Y_0$ is a quotient of $Y_0/v_1Y_0 = Y_0/Y_1 \subseteq Y/Y_1 = X_1$, which is finite. Thus $Y_0$ is finitely generated by Nakayama’s lemma. Since $Y/Y_0 = X_0$ is finite, $Y$ must be finitely generated. To see that $Y$ is torsion, we write

$$Y \sim Y_0 \sim \Lambda^r \bigoplus_{i=1}^s \Lambda/(p^{m_i}) \bigoplus_{j=1}^t \Lambda/(f_j(T)^{l_j}).$$

Notice that $Y_0/v_nY_0 \subseteq Y/v_nY_n = Y/Y_n$ is finite while $\Lambda/(v_n)$ is infinite, so $r = 0$ and $Y$ is a torsion module. To show that $X$ is a noetherian torsion module, by the following exact sequence

$$0 \to Y \to X \to \mathbb{Z}_p \to 0,$$  \hfill (1)

it suffices to show that $\mathbb{Z}_p$ is noetherian and torsion. This is easy to see since $\mathbb{Z}_p \cong \Lambda/TA$ and $\mathbb{Z}_p$ is generated by the unit element $1$. \qed

**Lemma 2.7** For any geometric $\mathbb{Z}_p$-extension of $k_\infty/k$ in which only finitely many primes of $k$ are ramified (not necessary totally ramified), let $L$ be the maximal unramified $p$-extension of $k_\infty$. Then both $X = \text{Gal}(L/k_\infty)$ and $Y = \text{Gal}(L/k_\infty \tilde{F}_q)$ are noetherian torsion $\Lambda$-modules, and there exists $e \geq 0$ such that

$$X_n \cong Y/v_{n,e}Y, \text{ for all } n \geq e.$$

**Proof.** Choose some $e$ such that all ramified primes are totally ramified in $k_\infty/k_e$. Then replace $k$ by $k_e$ in Theorem 2.6. \qed
Theorem 2.8 Let $k_{\infty}/k$ be a geometric $\mathbb{Z}_p$-extension with only finitely many ramified primes of $k$. Let $p^n$ be the exact power of $p$ dividing the class number of $k_n$. Then there exist integers $\lambda \geq 0$ and $\mu \geq 0$ and some $\nu$ such that

$$e_n = \lambda n + \mu p^n + \nu$$

for all $n \gg 0$ where $\lambda$ and $\mu$ are just the Iwasawa invariants of $\Lambda$-module $Y = \text{Gal}(\tilde{L}/k_{\infty})$.

Proof. Assume $Y \sim E$ where $E$ is an elementary torsion $\Lambda$-module. By the IW4, one has

$$p^n = |X_n| = |Y/E||E/v_{n,e}E| = (\text{const})|E/v_{n,e}E| = p^{\lambda n + \mu p^n + \nu}$$

for all $n \gg 0$.

The rest of the theorem follows from IW3. \qed

Now let us deal with the same problem in the case of constant $\mathbb{Z}_p$-extension.

Lemma 2.9 Let $L$ be the maximal unramified abelian $p$-extension of $k\tilde{\mathbb{F}}_q$. Let $X = \text{Gal}(L/k\tilde{\mathbb{F}}_q)$ and $G = \text{Gal}(L/k)$. Then

$$G = \Gamma'X,$$

where $\Gamma'$ is the subgroup of $G$ generated by some $\sigma \in G$ restricting to the Frobenius automorphism on $k\tilde{\mathbb{F}}_q$.

Proof. Let $\Gamma = \text{Gal}(k\tilde{\mathbb{F}}_q/k)$. For any $g \in G$ let $g|_{k\tilde{\mathbb{F}}_q} = (\sigma|_{k\tilde{\mathbb{F}}_q})^t$ for some $t$. Then $x = \sigma^{-t}g \in X$ and therefore $g = \sigma^t x \in \Gamma'X$. The opposite inclusion $\Gamma'X \subseteq G$ is obvious. \qed

Noticing that the action of $\Gamma$ on $G$ can be realized by extending it to $\Gamma'$, one can identify $\Gamma$ with $\Gamma'$. This lemma can be used in the proof of Lemma 2.4 and Lemma 2.5 in constant $\mathbb{Z}_p$-extension case. The rest of the discussion is exactly same. Hence one has the following theorem.

Theorem 2.10 Let $k_{\infty}/k$ be the constant $\mathbb{Z}_p$-extension of $k$ and $L$ be the maximal unramified abelian $p$-extension of $k_{\infty}$. Let $p^n$ be the exact power of $p$ dividing the class number of $k_n$. Then there exist integers $\lambda \geq 0$ and $\mu \geq 0$ and some $\nu$ such that

$$e_n = \lambda n + \mu p^n + \nu$$

for all $n \gg 0$ where $\lambda$ and $\mu$ are just the Iwasawa invariants of $\Lambda$-module $X = \text{Gal}(L/k_{\infty})$. 9
Later on, for a constant $\mathbb{Z}_p$-extension we will see that $\mu = 0$ as is well-known.

**Definition.** If $k_\infty/k$ is a $\mathbb{Z}_p$-extension and $h_n = p^{\lambda n + \mu n + \nu}$ for all sufficiently large $n$. We denote the Iwasawa invariants by $\lambda(k_\infty/k) = \lambda$ and $\mu(k_\infty/k) = \mu$.

Now will discuss some relations among Iwasawa invariants of noetherian torsion $\Lambda$-modules. For any noetherian $\Lambda$-module $X$, denote its $p$-torsion part by $X_p$. It is easy to see that there is some $m > 0$ such that $p^m x = 0$ for all $x \in X_p$ since $X \sim E$ for some elementary $E$.

**Lemma 2.11** If $X \sim Y$ are noetherian torsion $\Lambda$-modules then $X_p \sim Y_p$.

**Proof.** Let $\varphi$ be a homomorphism from $X$ to $Y$ with finite kernel and cokernel, let $\varphi|_{X_p}$ be the restriction of $\varphi$ to $X_p$. Obviously $\varphi|_{X_p}$ has finite kernel. We need to show $Y_p/\varphi(X_p)$ is finite. Since $Y/\varphi(X)$ is finite, from the following exact sequence

$$0 \longrightarrow Y_p \cap \varphi(X)/\varphi(X_p) \longrightarrow Y_p/\varphi(X_p) \longrightarrow Y/\varphi(X)$$

it only remains to show that $Y_p \cap \varphi(X)/\varphi(X_p)$ is finite. Choose a positive integer $m$ such that $p^m$ annihilates $X_p$ and $Y_p$. Let $X' = \{x \in X : \varphi(x) \in Y_p\}$. Then

$$\varphi(p^m X') = p^m \varphi(X') \subset p^m Y_p = \{0\}.$$

Thus $p^m X' \subseteq \ker(\varphi)$ which is a finite set. Let

$$p^m X' = \{a_i = p^m x_i : 1 \leq i \leq t\}.$$

For any $x \in X'$ there exists some $i$ such that $p^m x = a_i = p^m x_i$ thus $x - x_i \in X_p$ and

$$\varphi(x) = \varphi(x - x_i) + \varphi(x_i) \in \varphi(X_p) + \varphi(x_i).$$

Therefore the finite set $\{\varphi(x_i) : 1 \leq i \leq t\}$ generates $\varphi(X) \cap Y_p/\varphi(X_p)$ which is what we wanted to show. \(\square\)

Let $M$ be a noetherian torsion $\Lambda$-module. Let $M^\Gamma = \{m \in M : Tm = 0\}$ and $M_\Gamma = M/TM$. Define

$$Q(M) = |M^\Gamma|/|M_\Gamma|.$$

Then the following results are well-known:
Q1. If $M$ is finite, then $Q(M) = 1$.

Q2. If the sequence of $\Lambda$-modules

$$0 \rightarrow A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_m \rightarrow 0$$

is exact then $\prod_{i=1}^{m} Q(A_i)^{(-1)^i} = 1$.

Q3. If $X \sim Y$ then $Q(X) = Q(Y)$.

Lemma 2.12 Let $X$ be a noetherian torsion $\Lambda$-module and $\mu(X)$ its Iwasawa $\mu$-invariant. Then

$$Q(X_p) = p^{-\mu(X)}.$$

Proof. Assume that $X$ is pseudo-isomorphic to an elementary $\Lambda$-module $E$ with

$$E = \bigoplus_{i=1}^{s} \Lambda/(p^{m_i}) \oplus \bigoplus_{j=1}^{t} \Lambda/(f_j(T)^{l_j}),$$

where $s, t, m_i, l_j \in \mathbb{Z}$, $f_j$ is distinguished and irreducible. Then, from Lemma 2.11 one has

$$X_p \sim \bigoplus_{i=1}^{s} \Lambda/(p^{m_i}).$$

If $M = \Lambda/(p^{m_j})$, then $|M| = 1$ and $|M| = p^{m_j}$, so $Q(\Lambda/(p^{m_j})) = p^{-m_j}$ and therefore $Q(X_p) = Q(E_p) = p^{-\sum m_j} = p^{-\mu(X)}$ by Q3. The proof of the lemma is complete. □

Proposition 2.13 If one has a noetherian torsion $\Lambda$-module exact sequence

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

then $\lambda(Y) = \lambda(X) + \lambda(Z)$ and $\mu(Y) = \mu(X) + \mu(Z)$.

Proof. Let $\varphi$ be a homomorphism from $Y$ to $E$ with finite kernel and cokernel where $E$ is elementary. From the following diagram

$$
\begin{array}{ccccccccc}
0 & \rightarrow & X & \rightarrow & Y & \rightarrow & Z & \rightarrow & 0 \\
& & \varphi|_X & & \varphi & & \varphi & & \\
0 & \rightarrow & \varphi(X) & \rightarrow & E & \rightarrow & E/\varphi(X) & \rightarrow & 0
\end{array}
$$

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and the Snake lemma, one has an exact sequence

\[ 0 \rightarrow \ker(\varphi|_X) \rightarrow \ker(\varphi) \rightarrow \ker(\tilde{\varphi}) \rightarrow 0 \rightarrow E/\varphi(Y) \rightarrow (E/\varphi(X))/\tilde{\varphi}(Z) \rightarrow 0. \]

Since \( \ker(\varphi) \) and \( E/\varphi(Y) \) are finite every term in the sequence above is finite. Therefore we can reduce our discussion to the case where \( Y \) is an elementary \( \Lambda \)-module. Choose some \( m > 0 \) such that \( p^m \) annihilates \( X_p, Y_p \) and \( Z_p \).

From the following diagram

\[
\begin{array}{ccc}
0 & \rightarrow & X \\
\downarrow{p^m} & & \downarrow{p^m} \\
0 & \rightarrow & Y \\
\downarrow{p^m} & & \downarrow{p^m} \\
& & \\
0 & \rightarrow & Z \\
\end{array}
\]

and the Snake lemma, one has the following exact sequence

\[ 0 \rightarrow X_p \rightarrow Y_p \rightarrow Z_p \rightarrow X/p^m X \xrightarrow{f} Y/p^m Y \rightarrow Z/p^m Z \rightarrow 0. \]

If we identify \( X \) with a submodule of \( Y \), then \( \ker(f) = p^m Y \cap X/p^m X \). Let \( S = p^m Y \cap X \subseteq p^m Y \). Since \( p^m Y \) is finitely generated free \( \mathbb{Z}_p \)-module, so is \( S \). Therefore \( \ker(f) = S/p^m X \subseteq S/p^m S \) is finite. Thus \( Q(\ker(f)) = 1 \) by Q1. From the exact sequence

\[ 0 \rightarrow X_p \rightarrow Y_p \rightarrow Z_p \rightarrow \ker(f) \rightarrow 0 \]

one has \( Q(Y_p) = Q(X_p)Q(Z_p) \) by Q2. Hence \( \mu(Y) = \mu(X) + \mu(Z) \) follows from Lemma 2.12.

Noticing that \( \lambda(M) = \dim_{\mathbb{Q}_p}(M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \) one also has \( \lambda(Y) = \lambda(X) + \lambda(Z) \) by tensoring the exact sequence

\[ 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0 \]

with \( \mathbb{Q}_p \) and then counting dimensions. \( \square \)

**Corollary 2.14** Let \( k_\infty/k \) be a \( \mathbb{Z}_p \)-extension (geometric or constant), and let \( L \) be the maximal unramified \( p \)-extension of \( k_\infty \), \( X = \text{Gal}(L/k_\infty) \). Then \( \mu(X) = \mu(k_\infty/k) \) and \( \lambda(X) = \lambda(k_\infty/k) + d \), \( d = 0 \) if the extension is constant, \( d = 1 \) if the extension is geometric.
Proof. If $k_\infty/k$ is a constant field extension then the result follows from Theorem 2.10. If $k_\infty/k$ is geometric then it follows from the exact sequence (1) in the proof of Theorem 2.6, Theorem 2.8 and the above proposition that

$$\mu(X) = \mu(k_\infty/k) + \mu(\mathbb{Z}_p)$$

$$\lambda(X) = \lambda(k_\infty/k) + \lambda(\mathbb{Z}_p).$$

But $\mu(\mathbb{Z}_p) = 0$ and $\lambda(\mathbb{Z}_p) = 1$ since $\mathbb{Z}_p \cong \Lambda/(T)$. This finishes the proof. □

3 Unboundedness of Iwasawa $\mu$-invariant in $\mathbb{Z}_p$-extensions

In this section we will prove the following

**Theorem 3.1** For any $N > 0$ and any global function field $k$, there exist infinitely many cyclic extensions $k'/k$ of degree $p$ which has a geometric $\mathbb{Z}_p$-extension $k'_\infty/k'$ with Iwasawa $\mu$-invariant $\mu(k'_\infty/k') > N$.

We need several preliminary results.

**Lemma 3.2** Let $L/k$ be geometric cyclic extension of function fields of degree $n$. Let $G = \text{Gal}(L/k)$ and $J_L = C_0(L)$. As usual let $J^G_L$ be the set of elements of $J_L$ fixed by $G$ and $e_1, e_2, \ldots, e_t$ be the ramification indices for the $t$ ramified primes in $L/k$. Then

$$n(q-1)|J^G_L| = h_K e_1 e_2 \ldots e_t c(L/k),$$

where $c(L/k) \leq [L : k](q-1)$ is some constant integer depending on $L$ and $k$.

**Proof.** See [24, Theorem 8]. □

**Lemma 3.3** Let $k$ be a field complete with respect to a discrete valuation and of characteristic $p$ with perfect residue class field. Let $t$ be a uniformizing parameter. Let $k'$ be the extension of $k$ obtained by joining all roots of $x^p - x = a$ to it where $a = \sum_{-m}^\infty c_v t^v$ for some $m > 0$, $p \nmid m$ and $c_{-m} \neq 0$. Then the different of $k'/k$ is given by

$$D(k'/k) = p^{(m+1)(p-1)},$$

where $p$ is the only prime of $k'$.
Proof. See [1, Chapter 10 §4, pp. 203–205]. □

Lemma 3.4 Let $k$ be a global function field. There exists a $\mathbb{Z}_p$-extension of $k$ in which an infinite number of primes of $k$ are completely decomposed.

Proof. [7, Theorem 3]. □

Proof of Theorem 3.1. By Lemma 3.4, there exists a $\mathbb{Z}_p$-extension $k_\infty/k$ in which an infinite number of primes of $k$ are completely decomposed. Let $\{P_i : i \geq 0\}$ denote the set of these primes. Take an arbitrary positive integer $t$. By the Chinese Remainder Theorem we can find $f_i \in A$ such that $\text{ord}_{P_i}(f_j) = \delta_{ij}$ for all $1 \leq i, j \leq t$ where $\delta_{ij}$ is the Kronecker symbol. Let $k'_n$ be the extension of $k_n$ obtained by joining to it all roots of $x^p - x = f_1 f_2 \cdots f_t$.

Clearly $k' = k'_0$ is a cyclic extension of degree $p$ over $k$. By Lemma 3.3, one sees that $P_1, \ldots, P_t$ are ramified in $k'/k$. Thus all the primes of $k_n$ lying above $P_1, \ldots, P_t$ are ramified in $k'_n/k_n$. The number of such primes of $k_n$ is $tp^n$ since $P_1, \ldots, P_t$ are completely decomposed in $k_n/k$. Since all the ramification indices are equal to $p$, applying Lemma 3.2 one has

$$p(q - 1)|J_{k'_n}^{\mathbb{Z}_p} = h_{k_n} p^{tp^n + sc(k'_n/k_n)},$$

where $c(k'_n/k_n)$ is an integer and $s$ is the number of other ramified primes of $k'_n$. Let $X'_n$ be the $p$-primary subgroup of the class group of $k'_n$. Considering the $p$-divisibility of the above identity one has $p^{(p^n - 1)}|X'_n|$. Thus $e_n \geq tp^n - 1$ and $\mu \geq t$ for the extension $k'_n/k'_0$. Noticing that the integer $t$ is arbitrary, we can find cyclic extension $k'/k$ of degree $p$ and a $\mathbb{Z}_p$-extension $k'_\infty$ such that $t_1 = \mu(k'_\infty/k') > N$. Now for the same reason we can find a series of cyclic extensions $k^{(i)}/k(i) \geq 1)$ of degree $p$ and geometric $\mathbb{Z}_p$-extensions $k^{(i)}/k(i)$ such that $\mu(k^{(i+1)}/k^{(i+1)}) > \mu(k^{(i)}/k^{(i)})$. This concludes our proof of Theorem 3.1. □

4 Class numbers in $\mathbb{Z}_d^p$-extensions

In this section we let $K_\infty/k$ be a $\mathbb{Z}_d^p$-extension satisfying assumption FRP with Galois group $\Gamma$. Let $K_n$ be the fixed field of $\Gamma^n$. Let $p^n$ be the $p$-part
of the class number of $K_n$. We want to know what $e_n$ will look like when $n$ gets very large. For all the results in this section we only sketch the outline of the proofs.

Let $L$ be the maximal unramified abelian $p$-extension of $K_\infty$ and $X = \text{Gal}(L/K_\infty)$. Then $X$ can be considered as a $\mathbb{Z}_p$-module on which $\Gamma$ acts through $G = \text{Gal}(L/k)$ in a natural way since $X$ is a normal abelian subgroup of $G$. Let $\Lambda_d$ denote the complete group ring of $\Gamma = \langle \sigma_1, \ldots, \sigma_d \rangle$ over $\mathbb{Z}_p$. Take $T_i = \sigma_i - 1$ for $1 \leq i \leq d$, then it easy to see that $\Lambda_d \cong \mathbb{Z}_p[[T_1, \ldots, T_d]]$. Once we have fixed the topological generators of $\Gamma$ we can regard $X$ as a $\mathbb{Z}_p[[T_1, \ldots, T_d]]$-module.

**Theorem 4.1** $X$ is a noetherian torsion $\Lambda_d$-module.

**Proof.** The proof goes by induction on $d$ similar to the one for [8, Theorem 1]. The first step of the induction is Theorem 2.6. Note that the assumption that there are only finitely many ramified primes in $K_\infty/k$ is essential so that the argument on top of [8, p. 207] can be applied to our situation. □

**Theorem 4.2** For $d \geq 2$, there exist integers $m_0$, $l_0$ and a real number $\alpha$ determined by $X$ such that

$$e_n = (m_0 p^n + l_0 n + \alpha)p^{(d-1)n} + O(n p^{(d-2)n}).$$

Furthermore $\alpha$ is rational if $d = 2$.

**Proof.** See [22, Theorem 3.13]. For the most part the proof relies on purely module theoretic results and Theorem 4.1. No properties special to number fields are essentially used. □

**Theorem 4.3** If $X$ is finite as $\mathbb{Z}_p$-module, let $\lambda = \text{rank}(X)$. Then there exist integer $\nu$ such that

$$e_n = \lambda n + \nu \quad \text{for all } n \gg 0.$$

**Proof.** See [4, Theorem 2]. □

**Theorem 4.4** If $L/k$ is a $\mathbb{Z}_p$-extension inside $K_\infty/k$. Then $\mu(L/k) = m_0$ provided $L$ lies outside of finitely many $\mathbb{Z}_p^{d-1}$-extensions of $k$. 15
Proof. See [19, Theorem I]. □

In classical setting, Cuoco considered the relations between invariants of \( \mathbb{Z}_p^{d-1} \)-extensions inside \( \mathbb{Z}_p^{d} \)-extension and those of the \( \mathbb{Z}_p^{d} \)-extension. We can also obtain the following result

**Theorem 4.5** Let \( E \) be a \( \mathbb{Z}_p^{d-1} \)-extension of \( k \) and \( k_{\infty} \) a \( \mathbb{Z}_p \)-extension of \( k \) such that \( K_{\infty} = Ek_{\infty} \) and \( E \cap k_{\infty} = k \). Let \( k_n \) be the unique subfield of \( k_{\infty} \) of degree \( p^n \) over \( k \). Then there are constants \( l, m_1, c, c_1 \), which are independent of both \( k_{\infty} \) and \( n \), such that for all sufficiently large \( n \),

\[
m_0(Ek_n/k_n) = m_0(K_{\infty}/k)p^n + m_1n + c_1, \quad l_0(Ek_n/k_n) = lp^n + c.
\]

Furthermore, \( m_1 \leq l_0(K_{\infty}/k) \).

Proof. The proof of this theorem is almost the same as the one in number field case under the assumption FRP. One need use Theorem 4.1 the proof of which depends on the assumption FRP crucially. For details we refer the interested readers to the proof of [3, Theorem 1]. □

Remark. The following question corresponding to [19, Theorem II] is open in function field case: Fix base field \( k \), are the Iwasawa \( \mu \)-invariants bounded for all the \( \mathbb{Z}_p \)-extensions over \( k \)? The reason it is unsolved is that for any global function field there are infinitely many \( \mathbb{Z}_p \)-extensions over it. However, if we allow the base field to change then Theorem 3.1 shows that the \( \mu \)-invariants are unbounded.

## 5 Iwasawa invariants over \( \mathbb{Z}_p^2 \)-extensions

Let \( k_{\infty} \) and \( k'_{\infty} \) be two \( \mathbb{Z}_p \)-extensions of \( k \) (which are constant or geometric extensions) such that \( k_{\infty} \cap k'_{\infty} = k \). If we put \( K_n = k_nk'_{\infty} \) where \( k_n \) is the \( n \)-th layer of \( k_{\infty} \) one sees that \( K_n \) is a \( \mathbb{Z}_p \)-extension of \( k_n \) and therefore we can speak of the Iwasawa invariants \( \lambda_n = \lambda(K_n/k_n) \) and \( \mu_n = \mu(K_n/k_n) \). In this section we will discuss how these invariants change when \( n \) increases. By using Rosen’s results we will make the following theorem (essentially due to Cuoco) more precise when \( k_{\infty} \) is a constant \( \mathbb{Z}_p \)-extension of \( k \).
Theorem 5.1 Let $K = k_{\infty}k'_{\infty}$ and assume that there are only finitely many primes of $k$ which are ramified in $K/k$. Then there are constants $l$, $m_0$, $m_1$, $c$ and $c_1$, which are independent of $n$, such that for all sufficiently large $n$

$$\lambda_n = lp^n + c, \mu_n = m_0p^n + m_1n + c.$$ 

Furthermore, $m_0 = m_0(K/k)$ is independent of choices of $k_{\infty}$ and $k'_{\infty}$.

Proof. This is the special case of Theorem 4.5. □

Now we will study the case when one of $k_{\infty}$ and $k'_{\infty}$ is a constant $\mathbb{Z}_p$-extension. First, we state some results from [23] and [5]. From algebro-geometric point of view $k$ can be realized as the field of $F_q$-rational functions of a complete non-singular curve $C$. The Jacobian $J$ of $C$ is an abelian variety of dimension $g_k$, the genus of $k$. The degree zero divisor class group $C_0(k)$ of $k$ is isomorphic to $J(F_q)$, the group of $F_q$-rational points on $J$. Let $F_n$ denote the extension over $F_q$ of degree $p^n$ and $F_{\infty} = \cap_n F_n$. For any field $E$ let $J(E)(m)$ and $J(E)(p)$ denote the points of order $m$ and $p$-power in $J(E)$ respectively. Put $J_p = J(\bar{F}_q)$, where $\bar{F}_q$ is the algebraic closure of $F_q$. Let $F_q(J_p)$ be the field obtained by adjoining the coordinates of the points of $J_p$ to $F_q$.

Definition. We say that $p$ is regular for $k$ if $p \nmid [F_q(J_p) : F_q]$. A finite abelian $p$-group $A$ is said to have type $(m_1, \ldots, m_r)$ ($m_i > 0$) if $A \cong \mathbb{Z}/(p^{m_1}) \times \cdots \times \mathbb{Z}/(p^{m_r})$.

Let $r_k$ be the $p$-rank of $J(F_{\infty} \cap F_q(J_p))$. It is easy to see that if $p$ is regular then $r_k = p$-rank($J(\bar{F}_q)) = p$-rank($C_0(k)$). Put

$$p' = \begin{cases} 4 & \text{if } p = 2, \\ p & \text{if } p > 2. \end{cases}$$

If $p$ is irregular for $k$ let $F_{\delta(p)} = F_{\infty} \cap F_q(J_{p'})$. If $p$ is regular for $k$ let $\delta(p) = 1$ if $p = 2$ and $\delta(p) = 0$ if $p > 2$.

Lemma 5.2 Let $(m_1, \ldots, m_r)$ be the type of $J(F_{\delta(p)})(p)$. Then the type of $J(F_j)(p)$ is $(m_1 + j - \delta(p), \ldots, m_r + j - \delta(p))$ for all $j \geq \delta(p)$.

Proof. See [23, p. 293]. □

Remark. In the above lemma $r_p \geq r_k$ and the equality holds when $p > 2$. Also $r_p$ depends only on $k$, so we will write $r_p(k)$ if we want to emphasize this.
Lemma 5.3 Let $K = \tilde{k}\tilde{F}_q$. Let $k'$ be a geometric extension of $k$ of degree $p$ and $L = k'\tilde{F}_q$. Then $r_p(k)$ (resp. $r_p(k')$) is the $p$-rank of the class group of $K$ (resp. $L$). Moreover

$$r_p(k) - 1 = p(r_p(k') - 1) + (p - 1)t$$

where $t$ is the number of ramified primes of $L$ in $L/K$.

Proof. The first statement follows directly from Lemma 5.2 and the second is a special case of [5, (1.1)].

Proposition 5.4 The Iwasawa invariants $\lambda(k\tilde{F}_q/k) = r_p$ and $\mu(k\tilde{F}_q/k) = 0$.

Proof. Let $k_n = kF_n$ be the $n$-th layer of the constant $\mathbb{Z}_p$-extension $k\tilde{F}_q/k$. By Lemma 5.2, when $n$ is sufficiently large the $p$-primary part of $C_0(k_n)$ has $p$-rank $r_p$ and its order is increased by $r_p$ when $n$ is increased by 1. The proposition follows at once.

Now we need zeta function of $k$ in our computation of the rank $r_p$. The zeta function is defined by

$$\zeta_k(s) = \prod_{P: \text{prime of } k} (1 - N_P^{-s})^{-1},$$

where the norm $N_P = q^{\deg P}$. Let $g$ be genus of $k$. It is well known that $\zeta_k(s)$ is a rational function in $u = q^{-s}$ and can be written as

$$\zeta_k(s) = \frac{L_k(s)}{(1-u)(1-qu)}, \quad (2)$$

where $L(u) = \prod_{i=1}^{2g}(1 - w_i u)$ is a polynomial of degree $2g$ with rational integer coefficients. It is a standard fact that $L(1)$ is the class number of $k$. Moreover the $w_i$’s are algebraic integers and $w_i \to qu_i^{-1}$ is a permutation of these numbers. One very interesting fact about $L_k(u)$ is that if $E/k$ is a constant field extension of degree $m$ then

$$L_E(u) = \prod_{i=1}^{2g}(1 - w_i^m u).$$

Proposition 5.5 Let $\zeta_k(s)$ be the zeta function of $k$ as in (2). Then $r_p$ is the number of $\omega_i$’s such that $\omega_i \equiv 1 \pmod{\wp}$, where $\wp$ is any fixed prime of $\mathbb{Q}(\omega_1, \ldots, \omega_{2g})$ lying above $p \in \mathbb{Z}$.
**Proof.** Let \( k_n \) be the constant field extension of \( k \) of degree \( p^n \). Then the zeta function of \( k_n \) is
\[
\zeta_{k_n}(s) = \prod_{i=1}^{2g} (1 - \omega_i^{p^n} u)/(1 - u)(1 - qu).
\]
Hence the class number of \( k_n \) can be expressed as
\[
h_n = \prod_{i=1}^{2g} (1 - \omega_i^{p^n}).
\]
Let \( \omega \) be one of the \( \omega_i \)'s above. Then one can write
\[
\omega^{p^n} - 1 = (1 + a)^{p^n} - 1
\]
\[
= a \left( C_{p^n}^1 + C_{p^n}^2 a + \cdots + C_{p^n}^{p^n-1} a^{p^n-2} + a^{p^n-1} \right).
\]
Let \( \wp \) be a fixed prime of \( \mathbb{Q}(\omega_1, \cdots, \omega^{2g}) \) lying above \( p \). If \( \wp \) does not divide \( a = \omega - 1 \) then \( \wp \) does not divide \( \omega^{p^n} - 1 \) either since \( p|C_i^{p^n} \) for \( 0 < i < p^n \).
In the algebraic closure \( \Omega_p \) of \( \mathbb{Q}_{p^n} \), \( \omega^{p^n} - 1 = \prod_{i=1}^{p^n} (\omega - \zeta_i) \) where the \( \zeta_i \) runs through the \( p^n \)-th root of unity. If \( \wp \) divides \( a = \omega - 1 \), then \( \text{ord}_p(a) > 1/(p^n - 1) \) for all \( m > n_0 = \log_p(1 + 1/\text{ord}_p(a)) \) and therefore
\[
\text{ord}_p(\omega - \zeta_i) = \text{ord}_p(a + 1 - \zeta_i) = \text{ord}_p(1 - \zeta_i)
\]
for all primitive \( m \)-th roots of unity. Thus the above holds with at most finitely many \( (< p^{n_0+1}) \) exceptions. Since \( p^n = \prod_{i=1}^{p^n-1} (1 - \zeta_i) \) one sees that there is a constant \( c \) independent of \( n \) such that
\[
\text{ord}_p(\omega^{p^n} - 1) = n + c \quad \text{for all} \ n \gg 0.
\]
Let \( \lambda \) be the number of \( \omega_i \)'s such that \( \omega_i \equiv 1 \pmod{\wp} \). Then \( \text{ord}_p(h_n) = \lambda n + \nu \) for some constant \( \nu \). Now we can finish the proof by combining this with Proposition 5.4.

Let \( k_\infty/k \) be a geometric \( \mathbb{Z}_p \)-extension of \( k \) with finitely many ramified primes. Let \( \lambda_n \) and \( \mu_n \) be the Iwasawa invariants for \( k_n \tilde{\mathbb{Q}}/k_n \). Then \( \mu_n = 0 \) for all \( n \) by Proposition 5.4. From Proposition 5.4 and Lemma 5.3 one has
\[
\lambda_n - 1 = p(\lambda_{n-1} - 1) + (p - 1)t,
\]
where \( t \) is the number of the primes of \( k_n \) ramified in \( k_n/k_{n-1} \). Choose some \( n_0 \) such that \( k/k_{n_0} \) is totally ramified, then \( t \) is independent of \( n \) when \( n \geq n_0 \). By solving the recurrence formula one has:
Proposition 5.6 Let \( k_\infty/k \) be a geometric extension of \( k \). Let and \( \lambda_n \) and \( \mu_n \) be the Iwasawa invariants of \( k_\infty \). Then

\[
\begin{align*}
\mu_n &= 0 \quad \text{for all } n \geq 0 \\
\lambda_n &= p^{n-n_0}(\lambda_{n_0} + t - 1) - t + 1 \quad \text{for all } n \geq n_0.
\end{align*}
\]

Theorem 5.7 Let \( k_\infty/k \) be a geometric \( \mathbb{Z}_p \)-extension with finitely many ramified primes. Then there exists some constant \( c \) such that for \( n \gg 0 \)

\[
\begin{align*}
\mu(k_\infty F_n/kF_n) &= \mu(k_\infty/k) + n c, \\
\lambda(k_\infty F_n/kF_n) &= \lambda(k_\infty/k).
\end{align*}
\]

Proof. Choose some \( m_0 \) such that all ramified primes in \( k_\infty/k_{m_0} \) are totally ramified. For the extension \( k_\infty/k \) one has

\[
e_m = \lambda m + \mu p^m + \nu \quad \text{for all } m \gg 0
\]

for some constants \( \lambda = \lambda(k_\infty/k) \), \( \mu = \mu(k_\infty/k) \) and \( \nu \). One knows that \( l_m = r_p(k_m) \) and \( l_{m_0} = r_p(k_{m_0}) \) are the \( \lambda \)-invariants of \( k_m \) and \( k_{m_0} \) respectively by Proposition 5.4. For the extension \( k_\infty F_n/kF_n \), by Lemma 5.2 one has

\[
\text{ord}_p(h(K_m)) = e_m + n l_m = \lambda m + \mu p^m + \nu + n l_m \quad \text{for all } n \gg 0
\]

where \( K_m \) is the \( m \)-th layer of \( k_\infty F_n/kF_n \) which is a constant extension of \( k_m \) of degree \( p^n \). Then Proposition 5.6 yields

\[
l_m = p^{m-m_0}(l_{m_0} + c' - 1) - c' + 1
\]

for some constant \( c' \). Take \( c = p^{-m_0}(l_{m_0} + c' - 1) \) one has

\[
\text{ord}_p(h(K_m)) = \lambda m + \mu p^m + \nu + n r_m = \lambda m + (\mu + n c)p^m + \nu - c' + 1.
\]

This shows that for \( n \gg 0 \)

\[
\begin{align*}
\mu(k_\infty F_n/kF_n) &= \mu(k_\infty/k) + n c, \\
\lambda(k_\infty F_n/kF_n) &= \lambda(k_\infty/k)
\end{align*}
\]

as desired. \( \square \)

Corollary 5.8 In Theorem 5.1, if the \( \mathbb{Z}_p^2 \)-extension \( K \) contains a constant \( \mathbb{Z}_p \)-extension then \( m_0 = 0 \). If \( k_\infty/k \) is constant \( \mathbb{Z}_p \)-extension then \( l = 0 \).
Proof. Note that $m_0$ depends only on $K$ by Theorem 5.1.

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