

Iwasawa Theory of \mathbb{Z}_p^d -Extensions over Global Function Fields

Chaoqun Li and Jianqiang Zhao¹

Abstract. In this paper we study the Iwasawa theory of \mathbb{Z}_p^d -extensions of global function fields k over finite fields of characteristic p . When $d = 1$ we first show that Iwasawa invariants are well defined under the assumption that only finitely many primes are ramified in the extension, then we prove that the Iwasawa μ -invariant can be arbitrarily large for some extension of any given base field k . After giving some general results of \mathbb{Z}_p^d -extensions we finally study the behaviors of Iwasawa invariants in \mathbb{Z}_p^2 -extensions in some detail, especially when the extensions contain a constant \mathbb{Z}_p -extension. The whole paper is expository in nature though it contains some new results.

1 Introduction

Around 1940 A. Weil developed a theory which relates the zeta function of the function fields to the characteristic polynomials of Frobenius acting on p -power order torsion in the Jacobian of the corresponding curve. He chose p to be any prime different from the characteristic of the field. In looking for an analogue of this theory for number fields, beginning in late 1950's, K. Iwasawa carried out the fundamental work of what is now called the Iwasawa theory in a series of papers. To get a better analogy between p -power order torsion in the Jacobian on the function field side and the p -Sylow subgroup of the class group on the number field side, Iwasawa considered field tower obtained by adjoining the p -power roots of unity or more generally \mathbb{Z}_p -extensions for some fixed p . By definition, a \mathbb{Z}_p -extension of a field k is an infinite Galois extension k_∞/k with the Galois group isomorphic to the additive group of p -adic integers \mathbb{Z}_p . It is possible to regard a \mathbb{Z}_p -extension as a sequence of fields

$$k = k_0 \subset k_1 \subset \cdots \subset k_\infty = \bigcup_{n=1}^{\infty} k_n$$

¹Please send correspondence to the second author.

Email address: jzhao@math.brown.edu

with $\text{Gal}(k_n/k) \cong \mathbb{Z}/p^n\mathbb{Z}$. We say k_n is the n -th layer of the \mathbb{Z}_p -extension. Iwasawa at first named them Γ -extensions and later on he and others switched to the now standardized name “ \mathbb{Z}_p -extensions”. One of the prevailing objectives of Iwasawa theory is to study the behavior of the ideal class group of k_n as n varies.

Classically, there are several approaches to this problem: (A) via class number formula and p -adic L -functions; (B) via class number formula and p -adic measures on \mathbb{Z}_p ; (C) via the theory of noetherian modules over Iwasawa algebra $\varprojlim_{\mathbb{N}} \mathbb{Z}_p[\mathbb{Z}/p^n\mathbb{Z}]$. Approach (A) works particularly well for the so called cyclotomic \mathbb{Z}_p -extensions (i.e. the compositum of all the cyclotomic number fields with the base field). But it apparently fails for more general fields which do not possess a good class number formula. (B) was found by Sinnott (see [26] and [27]) when he studied the work of Ferrero and Washington. Neither (A) nor (B) can be used to handle the function field case because good analogues of p -adic L -functions and p -adic measures are still unborn for function fields. Approach (C) was the one initiated by Iwasawa and further developed by Serre, Greenberg, Cuoco, Monsky and many others. This method can be combined with class field theory and Kummer theory to get a lot of nice results.

The last approach of the above also has the merit of being generalizable to function field case, which is what we shall use in the present paper. More precisely we shall return to the function fields k of transcendental degree one over a finite field \mathbb{F}_q and deal with the situation when p is the characteristic of k . Our primary interest lies in the ideal class numbers of different layers in any \mathbb{Z}_p -extension over k . We are also concerned with \mathbb{Z}_p^d -extension K_∞/k such that the Galois group Γ is isomorphic to d copies of \mathbb{Z}_p . Let K_n be the fixed field of Γ^{p^n} . Then it is a standard fact that the group of divisor classes of K_n of degree zero $C_0(K_n)$ is finite. We will denote the p -part subgroup of $C_0(K_n)$ and its order by X_n and $h_n = p^{e_n}$ respectively. There is a rather complete Iwasawa theory in the number field case for $d = 1$. In this case, Iwasawa showed that there are constants μ , λ and ν such that $e_n = \lambda n + \mu p^n + \nu$. Cuoco made the first generalization of Iwasawa’s formula to $d = 2$ in his Ph.D thesis (see [2]). Exploiting Greenberg’s result (see [8]) about Λ_d -modules and Monsky’s result (see [18]) about d -variable power series over \mathbb{Z}_p , Cuoco and Monsky [4] were able to extend Iwasawa’s formula to all positive integers d .

Given a global function field k , an obvious \mathbb{Z}_p -extension is the constant field \mathbb{Z}_p -extension. If k_∞/k is such an extension, then there are no ramified

primes in k_∞ . In contrast to the constant extension, by a *geometric extension* we mean an extension which does not contain any constant field extension. A geometric \mathbb{Z}_p -extension of k is the one such that k_n/k is geometric for every n -th layer k_n . It is easy to see that any non-constant \mathbb{Z}_p -extension of k becomes a geometric \mathbb{Z}_p -extension by throwing away first finitely many layers. Thus a natural question is: how many geometric \mathbb{Z}_p -extensions does k have? The answer is a little surprising compared with the number field case: there are infinitely many. This can be derived from class field theory without too much difficulty (see [7, Theorem 3]).

Another difference between number field and function field \mathbb{Z}_p -extensions is that for a given \mathbb{Z}_p -extension of number field, there are only finitely many ramified primes in this extension. But in the function field case, there might be infinitely many primes of k which are ramified in a \mathbb{Z}_p -extension over k . This is not too hard to prove by using the ideas from [7]. Throughout this paper, we will assume that all \mathbb{Z}_p^d -extensions K_∞/k satisfies the following assumption:

FRP *Only finitely many primes of k ramify in K_∞ .*

If the assumption fails when $d = 1$ Gold and Kisilevsky ([7, Theorem 2]) showed essentially that one cannot define the Iwasawa invariants properly for the \mathbb{Z}_p -extension. In addition to this difficulty, while adapting the proofs of some key results in number fields to the function field \mathbb{Z}_p^d -extensions, we find that this assumption is actually indispensable (for instance, the proof of Theorem 4.1 and 5.1 would fail if the number of ramified primes of k is infinity.)

We organize this paper as follows. In §2, we give a brief argument about how h_n changes when n increases and define the Iwasawa μ -invariant and λ -invariant under the assumption **FRP**. Since the proofs are quite similar to the classical case we omit most of them and refer the reader to the very well written book by L. Washington [29] and other authors' papers on this topic.

In §3, we show that μ -invariant can be arbitrarily large for geometric \mathbb{Z}_p -extensions even under the assumption **FRP**. Similar result in number field case holds as showed by Iwasawa (see [13] or [14, p. 310]). Here we are unable to provide a good analogue of the following wonderful result in number field case: if k is abelian over \mathbb{Q} then the μ -invariant of any cyclotomic \mathbb{Z}_p -extensions over k vanishes ([6] or [26]). In spite of very good analogues of cyclotomic extensions ([10]) the Galois groups are too large to handle at present (see [9] and [16]).

In §4, we present some general results of \mathbb{Z}_p^d -extensions over k . As an application, in the last section we will discuss the growth rate of Iwasawa invariants when the base field k changes in another \mathbb{Z}_p -extension. Especially, if k_∞/k is a geometric \mathbb{Z}_p -extension with constant field \mathbb{F}_q and F_n is an extension of \mathbb{F}_q of degree p^n , then $k_\infty F_n/k F_n$ is also a geometric \mathbb{Z}_p -extension. Let λ_n and μ_n be the corresponding Iwasawa invariants. We will show that (Theorem 5.7) there exists some constant c such that for n sufficiently large $\lambda_n = \lambda(k_\infty/k)$ and $\mu_n = \mu_{n-1} + c$.

2 Iwasawa invariants in \mathbb{Z}_p -extensions

Let k_∞ be a geometric \mathbb{Z}_p -extension of k such that $k = k_0 \subset k_1 \subset \dots \subset k_n \subset \dots \subset k_\infty$, where k_n is the n -th layer of k_∞/k . In this section we will define the Iwasawa invariants under the assumption **FRP** in the introduction. Put $\Gamma = \text{Gal}(k_\infty/k) = \langle \gamma_0 \rangle$. Let L be the maximal unramified abelian p -extension of k_∞ . Let X and G be the Galois group of L/k_∞ and L/k respectively. Then one has the following diagram

$$\begin{array}{ccc} k_\infty & \xrightarrow{X} & L \\ \Gamma \downarrow & \nearrow G & \\ k & & \end{array}$$

One sees that $\Gamma \cong G/X$ and Γ acts on X by extending to G and makes X into a Γ -module. More precisely, let

$$0 \longrightarrow X \longrightarrow G \xrightarrow{\pi} \Gamma \longrightarrow 0$$

be a short exact sequence. For any $x \in X$ and any $r \in \Gamma$, choose any $g \in G$ such that $\pi(g) = r$, then define $x^r = gxg^{-1}$. It is easily to see that this is well-defined. In fact, let g_1 be another element of G such that $\pi(g_1) = r$, then $g^{-1}g_1 = x_0$ for some $x_0 \in X$. Notice that X is abelian, so

$$x^r = gxg^{-1} = gx_0^{-1}xx_0g^{-1} = g_1xg_1^{-1}$$

Let Λ be the completed group ring $\mathbb{Z}_p[[\Gamma]]$ which is isomorphic to $\mathbb{Z}_p[[T]]$ under the homomorphism sending $\gamma_0 - 1$ to T . As a projective limit of finite p -groups X is a compact Γ -module, which makes X into a compact Λ -module. We will show that X is noetherian Λ -torsion module and retrieve some information for X_n from some submodules of X .

Lemma 2.1 *Let L be the maximal unramified abelian p -extension of k_∞ , K_0 be the maximal unramified abelian p -extension of k contained in L . Let k_∞ be a geometric \mathbb{Z}_p -extension of k . Then there must be some prime of k which is ramified in k_∞/k .*

Proof. Suppose that k_∞ is a geometric \mathbb{Z}_p -extension of k such that k_∞ is unramified over k . Let $\tilde{\mathbb{F}}_q$ be the constant \mathbb{Z}_p -extension of k . Then clearly $k_\infty\tilde{\mathbb{F}}_q$ is unramified over k with Galois group $\mathbb{Z}_p \times \mathbb{Z}_p$. By the class field theory,

$$\text{Gal}(K_0/k) \cong \mathbb{Z}_p \times C_0(k).$$

One finds a contradiction by noticing that $C_0(k)$ is finite while $k_\infty\tilde{\mathbb{F}}_q \subseteq K_0$ by Lemma 2.2. \square

For a geometric \mathbb{Z}_p -extension, there are possibly infinitely many primes of k which are ramified in this extension. Now we assume that k_∞/k satisfies the hypothesis **FRP** in the introduction. The inertia groups of the ramified primes are subgroups of \mathbb{Z}_p which are of the form $p^d\mathbb{Z}_p$ for some d . Thus, we can choose n_0 such that all the ramified primes are totally ramified in k_∞/k_{n_0} . Hence, without loss of generality we can assume that ramified primes are totally ramified in k_∞/k .

Lemma 2.2 *Notation as in Lemma 2.1. Then K_0 is the maximal unramified abelian extension of $k\tilde{\mathbb{F}}_q$ contained in L .*

Proof. Let K'_0 be the maximal unramified p -abelian extension of $k\tilde{\mathbb{F}}_q$ such that $K_0 \subseteq K'_0 \subseteq L$. Then one has the following graph of field extensions

$$\begin{array}{ccccccc} k_\infty & \text{---} & k_\infty\tilde{\mathbb{F}}_q & \text{---} & k_\infty K'_0 & \text{---} & L \\ | & & | & & | & & / \\ k & \text{---} & k\tilde{\mathbb{F}}_q & \text{---} & K'_0 & & \end{array}$$

For any $\sigma \in \text{Gal}(L/k)$, $\sigma(K'_0)$ is the maximal unramified p -extension of $\sigma(k\tilde{\mathbb{F}}_q) = k\tilde{\mathbb{F}}_q$. Since the maximal unramified extension is unique, one has $\sigma(K'_0) = K'_0$, whence K'_0/k is Galois. Now k_∞/k is totally ramified and K'_0/k is unramified, so one has $\text{Gal}(K'_0/k) \cong \text{Gal}(K'_0 k_\infty/k_\infty)$, which is abelian by the definition of L and K'_0 . Thus $K_0 = K'_0$ as desired. \square

Now we know $X \cong \text{Gal}(L/k_\infty\tilde{\mathbb{F}}_q) \times \mathbb{Z}_p$ as abelian groups (see [7, Proposition 2]), where $\tilde{\mathbb{F}}_q$ is the constant \mathbb{Z}_p -extension of \mathbb{F}_q and $\mathbb{Z}_p \cong \text{Gal}(k_\infty\tilde{\mathbb{F}}_q/k_\infty)$.

It is easy to see that $k_\infty\tilde{\mathbb{F}}_q$ is a Galois extension over k since both k_∞/k and $k\tilde{\mathbb{F}}_q/k$ are Galois. Thus $\text{Gal}(L/k_\infty\tilde{\mathbb{F}}_q)$ is a normal subgroup of G and so is a submodule of X as Λ -module.

Lemma 2.3 *Let Γ_1 be the Galois group of $k_\infty\tilde{\mathbb{F}}_q/k\tilde{\mathbb{F}}_q$ and $Y = \text{Gal}(L/k_\infty\tilde{\mathbb{F}}_q)$ as a Λ -module. Then Y as a Γ_1 -module is isomorphic to Y as a Γ -module.*

Proof. Let $f : \sigma \rightarrow \sigma|_{k_\infty}$ be the restriction of Γ to Γ_1 . Define $G_1 = \text{Gal}(L/k\tilde{\mathbb{F}}_q)$. Then one has the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y & \longrightarrow & G_1 & \xrightarrow{\pi_1} & \Gamma_1 & \longrightarrow & 0 \\ & & \downarrow i & & \downarrow i & & \downarrow f & & \\ 0 & \longrightarrow & X & \longrightarrow & G & \xrightarrow{\pi} & \Gamma & \longrightarrow & 0. \end{array}$$

For any $y \in Y$ and $r \in \Gamma$, let g be an element of G such that $\pi(g) = r$, then $y^r = grg^{-1}$. Now let $r' = f^{-1}(r)$ and $g_1 \in G_1$ such that $\pi_1(g_1) = r'$. Then $\pi(g) = f(\pi_1(g_1)) = \pi(i(g_1)) = r$, hence $g^{-1}g_1$ is an element of X . But X is abelian, so

$$y^{r_1} = g_1 y g_1^{-1} = g_1 (g^{-1} g_1)^{-1} y g^{-1} g_1 g_1^{-1} = g y g^{-1} = y^r$$

as desired. □

From this lemma, it suffices to study Y as a Γ_1 -module. Let $\mathcal{P}_1, \dots, \mathcal{P}_s$ be the primes which are ramified in $k_\infty\tilde{\mathbb{F}}_q/k\tilde{\mathbb{F}}_q$. Let $I_i \subseteq G_1$ be the inertia group, then one has $I_i \cap Y = 1$ since $L/k_\infty\tilde{\mathbb{F}}_q$ is unramified. Now that $k_\infty\tilde{\mathbb{F}}_q/k\tilde{\mathbb{F}}_q$ is totally ramified at \mathcal{P}_i ,

$$I_i \hookrightarrow G_1/Y = \Gamma_1$$

is bijective since $L/k_\infty\tilde{\mathbb{F}}_q$ is maximal unramified. So

$$G_1 = I_i Y = Y I_i, \quad \text{for all } i = 1, \dots, s.$$

Let $\sigma_i \in I_i$ map to the generator γ_0 of Γ_1 . Then σ_i must be a topological generator of I_i and $\sigma_i = a_i \sigma_1$ for some $a_i \in Y$ since $I_i \subseteq Y I_1$. Notice that $a_1 = 1$.

Lemma 2.4 *Let G'_1 be the closure of the commutator subgroup of G_1 . Then*

$$G'_1 = Y^{\gamma_0-1} = TY$$

.

Proof. See [29, Lemma 13.14, p. 278]. □

Lemma 2.5 *Let Y_0 be the submodule of Y generated by $a_i, i = 2, \dots, s$ and by $Y^{\gamma_0^{-1}} = TY$. Let X_n be the p -part of $C_0(k_n)$ and $Y_n = v_n Y_0$, where*

$$v_n = 1 + \gamma_0^1 + \gamma_0^2 + \dots + \gamma_0^{p^n - 1} = \frac{(1 + T)^{p^n} - 1}{T}.$$

Then for all $n \geq 0$

$$X_n \cong Y/Y_n.$$

Proof. See [29, Lemma 13.15, p. 278]. □

Now we state some facts about Λ -modules.

IW1. ([29, Lemma 13.16, p. 279]) Let M be a compact Λ -module. Then M is finitely generated over Λ if and only if $M/(p, T)M$ is finite.

IW2. ([29, p. 271]) Two Λ -modules M and M' are said to be pseudo-isomorphic, denoted by $M \sim M'$, if there is a homomorphism $M \rightarrow M'$ with finite kernel and cokernel.

IW3. ([29, Theorem 13.12, p. 271]) If M is a finitely generated Λ -module. Then

$$M \sim \Lambda^r \oplus \bigoplus_{i=1}^s \Lambda/(p^{m_i}) \oplus \bigoplus_{j=1}^t \Lambda/(f_j(T)^{l_j})$$

where $r, s, m_i, l_j \in \mathbb{Z}$ and f_j are distinguished polynomials. Conventionally $\mu(M) = \sum_{i=1}^s m_i$ is called the Iwasawa μ -invariant and $\lambda(M) = \sum_{j=1}^t l_j \deg(f_j)$ is called the Iwasawa λ -invariant. Moreover we can make the f_j 's irreducible.

IW4. ([29, Proposition 13.21, Proposition 13.19, p. 283]) Let

$$v_{n,e} = 1 + \gamma_0^{p^e} + \gamma_0^{p^{2e}} + \dots + \gamma_0^{p^n - p^e} = v_n/v_e, \text{ for some } 0 < e < n.$$

If M is noetherian and torsion then there exists some constant ν such that

$$|M/v_{n,e}M| = p^{\mu p^n + \lambda n + \nu}, \text{ for all } n \gg 0.$$

Suppose Y and E are Λ -modules such that $Y \sim E$ and $Y/v_{n,e}Y$ is finite for all $n \geq e$. Then, there exists some constant ν such that

$$|Y/v_{n,e}Y| = p^\nu |E/v_{n,e}E| \quad \text{for all } n \gg 0.$$

Theorem 2.6 *Both Λ -modules $X = \text{Gal}(L/k_\infty)$ and $Y = \text{Gal}(L/k_\infty \tilde{\mathbb{F}}_q)$ are noetherian torsion Λ -module.*

Proof. In Lemma 2.5 we have seen that

$$X_n \cong Y/Y_n,$$

where X_n is the p -part of the class group of k_n , which is finite for all n . Since $v_1 = T \in (p, T)$, one sees that $Y_0/(p, T)Y_0$ is a quotient of $Y_0/v_1Y_0 = Y_0/Y_1 \subseteq Y/Y_1 = X_1$, which is finite. Thus Y_0 is finitely generated by Nakayama's lemma. Since $Y/Y_0 = X_0$ is finite, Y must be finitely generated. To see that Y is torsion, we write

$$Y \sim Y_0 \sim \Lambda^r \oplus \bigoplus_{i=1}^s \Lambda/(p^{m_i}) \oplus \bigoplus_{j=1}^t \Lambda/(f_j(T)^{l_j}).$$

Notice that $Y_0/v_nY_0 \subseteq Y/v_nY_0 = Y/Y_n$ is finite while $\Lambda/(v_n)$ is infinite, so $r = 0$ and Y is a torsion module. To show that X is a noetherian torsion module, by the following exact sequence

$$0 \rightarrow Y \rightarrow X \rightarrow \mathbb{Z}_p \rightarrow 0, \tag{1}$$

it suffices to show that \mathbb{Z}_p is noetherian and torsion. This is easy to see since $\mathbb{Z}_p \cong \Lambda/T\Lambda$ and \mathbb{Z}_p is generated by the unit element 1. \square

Lemma 2.7 *For any geometric \mathbb{Z}_p -extension of k_∞/k in which only finitely many primes of k are ramified (not necessary totally ramified), let L be the maximal unramified p -extension of k_∞ . Then both $X = \text{Gal}(L/k_\infty)$ and $Y = \text{Gal}(L/k_\infty \tilde{\mathbb{F}}_q)$ are noetherian torsion Λ -modules, and there exists $e \geq 0$ such that*

$$X_n \cong Y/v_{n,e}Y_e, \quad \text{for all } n \geq e.$$

Proof. Choose some e such that all ramified primes are totally ramified in k_∞/k_e . Then replace k by k_e in Theorem 2.6. \square

Theorem 2.8 *Let k_∞/k be a geometric \mathbb{Z}_p -extension with only finitely many ramified primes of k . Let p^{e_n} be the exact power of p dividing the class number of k_n . Then there exist integers $\lambda \geq 0$ and $\mu \geq 0$ and some ν such that*

$$e_n = \lambda n + \mu p^n + \nu \text{ for all } n \gg 0$$

where λ and μ are just the Iwasawa invariants of Λ -module $Y = \text{Gal}(L/k_\infty \tilde{\mathbb{F}}_q)$.

Proof. Assume $Y \sim E$ where E is an elementary torsion Λ -module. By the **IW4**, one has

$$\begin{aligned} p^{e_n} = |X_n| &= |Y/E| |E/v_{n,e}E| \\ &= (\text{const}) |E/v_{n,e}E| \\ &= p^{\lambda n + \mu p^n + \nu} \text{ for all } n \gg 0. \end{aligned}$$

The rest of the theorem follows from **IW3**. □

Now let us deal with the same problem in the case of constant \mathbb{Z}_p -extension.

Lemma 2.9 *Let L be the maximal unramified abelian p -extension of $k \tilde{\mathbb{F}}_q$. Let $X = \text{Gal}(L/k \tilde{\mathbb{F}}_q)$ and $G = \text{Gal}(L/k)$. Then*

$$G = \Gamma' X,$$

where Γ' is the subgroup of G generated by some $\sigma \in G$ restricting to the Frobenius automorphism on $k \tilde{\mathbb{F}}_q$.

Proof. Let $\Gamma = \text{Gal}(k \tilde{\mathbb{F}}_q/k)$. For any $g \in G$ let $g|_{k \tilde{\mathbb{F}}_q} = (\sigma|_{k \tilde{\mathbb{F}}_q})^t$ for some t . Then $x = \sigma^{-t} g \in X$ and therefore $g = \sigma^t x \in \Gamma' X$. The opposite inclusion $\Gamma' X \subseteq G$ is obvious. □

Noticing that the action of Γ on G can be realized by extending it to Γ' , one can identify Γ with Γ' . This lemma can be used in the proof of Lemma 2.4 and Lemma 2.5 in constant \mathbb{Z}_p -extension case. The rest of the discussion is exactly same. Hence one has the following theorem.

Theorem 2.10 *Let k_∞/k be the constant \mathbb{Z}_p -extension of k and L be the maximal unramified abelian p -extension of k_∞ . Let p^{e_n} be the exact power of p dividing the class number of k_n . Then there exist integers $\lambda \geq 0$ and $\mu \geq 0$ and some ν such that*

$$e_n = \lambda n + \mu p^n + \nu \text{ for all } n \gg 0$$

where λ and μ are just the Iwasawa invariants of Λ -module $X = \text{Gal}(L/k_\infty)$.

Later on, for a constant \mathbb{Z}_p -extension we will see that $\mu = 0$ as is well-known.

Definition. If k_∞/k is a \mathbb{Z}_p -extension and $h_n = p^{\lambda n + \mu p^n + \nu}$ for all sufficiently large n . We denote the Iwasawa invariants by $\lambda(k_\infty/k) = \lambda$ and $\mu(k_\infty/k) = \mu$.

Now will discuss some relations among Iwasawa invariants of noetherian torsion Λ -modules. For any noetherian Λ -module X , denote its p -torsion part by X_p . It is easy to see that there is some $m > 0$ such that $p^m x = 0$ for all $x \in X_p$ since $X \sim E$ for some elementary E .

Lemma 2.11 *If $X \sim Y$ are noetherian torsion Λ -modules then $X_p \sim Y_p$.*

Proof. Let φ be a homomorphism from X to Y with finite kernel and cokernel, let $\varphi|_{X_p}$ be the restriction of φ to X_p . Obviously $\varphi|_{X_p}$ has finite kernel. We need to show $Y_p/\varphi(X_p)$ is finite. Since $Y/\varphi(X)$ is finite, from the following exact sequence

$$0 \longrightarrow Y_p \cap \varphi(X)/\varphi(X_p) \longrightarrow Y_p/\varphi(X_p) \longrightarrow Y/\varphi(X)$$

it only remains to show that $Y_p \cap \varphi(X)/\varphi(X_p)$ is finite. Choose a positive integer m such that p^m annihilates X_p and Y_p . Let $X' = \{x \in X : \varphi(x) \in Y_p\}$. Then

$$\varphi(p^m X') = p^m \varphi(X') \subset p^m Y_p = \{0\}.$$

Thus $p^m X' \subseteq \ker(\varphi)$ which is a finite set. Let

$$p^m X' = \{a_i = p^m x_i : 1 \leq i \leq t\}.$$

For any $x \in X'$ there exists some i such that $p^m x = a_i = p^m x_i$ thus $x - x_i \in X_p$ and

$$\varphi(x) = \varphi(x - x_i) + \varphi(x_i) \in \varphi(X_p) + \varphi(x_i).$$

Therefore the finite set $\{\varphi(x_i) : 1 \leq i \leq t\}$ generates $\varphi(X) \cap Y_p/\varphi(X_p)$ which is what we wanted to show. \square

Let M be a noetherian torsion Λ -module. Let $M^\Gamma = \{m \in M : T m = 0\}$ and $M_\Gamma = M/TM$. Define

$$Q(M) = |M^\Gamma|/|M_\Gamma|.$$

Then the following results are well-known:

Q1. If M is finite, then $Q(M) = 1$.

Q2. If the sequence of Λ -modules

$$0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A_m \longrightarrow 0$$

is exact then $\prod_{i=1}^m Q(A_i)^{(-1)^i} = 1$.

Q3. If $X \sim Y$ then $Q(X) = Q(Y)$.

Lemma 2.12 *Let X be a noetherian torsion Λ -module and $\mu(X)$ its Iwasawa μ -invariant. Then*

$$Q(X_p) = p^{-\mu(X)}.$$

Proof. Assume that X is pseudo-isomorphic to an elementary Λ -module E with

$$E = \bigoplus_{i=1}^s \Lambda/(p^{m_i}) \oplus \bigoplus_{j=1}^t \Lambda/(f_j(T)^{l_j}),$$

where $s, t, m_i, l_j \in \mathbb{Z}$, f_j is distinguished and irreducible. Then, from Lemma 2.11 one has

$$X_p \sim \bigoplus_{i=1}^s \Lambda/(p^{m_i}).$$

If $M = \Lambda/(p^{m_j})$, then $|M^\Gamma| = 1$ and $|M_\Gamma| = p^{m_j}$, so $Q(\Lambda/(p^{m_j})) = p^{-m_j}$ and therefore $Q(X_p) = Q(E_p) = p^{-\sum m_j} = p^{-\mu(X)}$ by **Q3**. The proof of the lemma is complete. \square

Proposition 2.13 *If one has a noetherian torsion Λ -module exact sequence*

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

then $\lambda(Y) = \lambda(X) + \lambda(Z)$ and $\mu(Y) = \mu(X) + \mu(Z)$.

Proof. Let φ be a homomorphism from Y to E with finite kernel and cokernel where E is elementary. From the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow \varphi|_X & & \downarrow \varphi & & \downarrow \bar{\varphi} & & \\ 0 & \longrightarrow & \varphi(X) & \longrightarrow & E & \longrightarrow & E/\varphi(X) & \longrightarrow & 0 \end{array}$$

and the Snake lemma, one has an exact sequence

$$0 \rightarrow \ker(\varphi|_X) \rightarrow \ker(\varphi) \rightarrow \ker(\bar{\varphi}) \rightarrow 0 \rightarrow E/\varphi(Y) \rightarrow (E/\varphi(X))/\bar{\varphi}(Z) \rightarrow 0.$$

Since $\ker(\varphi)$ and $E/\varphi(Y)$ are finite every term in the sequence above is finite. Therefore we can reduce our discussion to the case where Y is an elementary Λ -module. Choose some $m > 0$ such that p^m annihilates X_p , Y_p and Z_p . From the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow p^m & & \downarrow p^m & & \downarrow p^m & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

and the Snake lemma, one has the following exact sequence

$$0 \longrightarrow X_p \longrightarrow Y_p \longrightarrow Z_p \longrightarrow X/p^m X \xrightarrow{f} Y/p^m Y \longrightarrow Z/p^m Z \longrightarrow 0.$$

If we identify X with a submodule of Y , then $\ker(f) = p^m Y \cap X/p^m X$. Let $S = p^m Y \cap X \subseteq p^m Y$. Since $p^m Y$ is finitely generated free \mathbb{Z}_p -module, so is S . Therefore $\ker(f) = S/p^m X \subseteq S/p^m S$ is finite. Thus $Q(\ker(f)) = 1$ by **Q1**. From the exact sequence

$$0 \longrightarrow X_p \longrightarrow Y_p \longrightarrow Z_p \longrightarrow \ker(f) \longrightarrow 0$$

one has $Q(Y_p) = Q(X_p)Q(Z_p)$ by **Q2**. Hence $\mu(Y) = \mu(X) + \mu(Z)$ follows from Lemma 2.12.

Noticing that $\lambda(M) = \dim_{\mathbb{Q}_p}(M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ one also has $\lambda(Y) = \lambda(X) + \lambda(Z)$ by tensoring the exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

with \mathbb{Q}_p and then counting dimensions. □

Corollary 2.14 *Let k_∞/k be a \mathbb{Z}_p -extension (geometric or constant), and let L be the maximal unramified p -extension of k_∞ , $X = \text{Gal}(L/k_\infty)$. Then $\mu(X) = \mu(k_\infty/k)$ and $\lambda(X) = \lambda(k_\infty/k) + d$, $d = 0$ if the extension is constant, $d = 1$ if the extension is geometric.*

Proof. If k_∞/k is a constant field extension then the result follows from Theorem 2.10. If k_∞/k is geometric then it follows from the exact sequence (1) in the proof of Theorem 2.6, Theorem 2.8 and the above proposition that

$$\begin{aligned}\mu(X) &= \mu(k_\infty/k) + \mu(\mathbb{Z}_p) \\ \lambda(X) &= \lambda(k_\infty/k) + \lambda(\mathbb{Z}_p).\end{aligned}$$

But $\mu(\mathbb{Z}_p) = 0$ and $\lambda(\mathbb{Z}_p) = 1$ since $\mathbb{Z}_p \cong \Lambda/(T)$. This finishes the proof. \square

3 Unboundedness of Iwasawa μ -invariant in \mathbb{Z}_p -extensions

In this section we will prove the following

Theorem 3.1 *For any $N > 0$ and any global function field k , there exist infinitely many cyclic extensions k'/k of degree p which has a geometric \mathbb{Z}_p -extension k'_∞/k' with Iwasawa μ -invariant $\mu(k'_\infty/k') > N$.*

We need several preliminary results.

Lemma 3.2 *Let L/k be geometric cyclic extension of function fields of degree n . Let $G = \text{Gal}(L/k)$ and $J_L = C_0(L)$. As usual let J_L^G be the set of elements of J_L fixed by G and e_1, e_2, \dots, e_t be the ramification indices for the t ramified primes in L/k . Then*

$$n(q-1)|J_L^G| = h_K e_1 e_2 \dots e_t c(L/k),$$

where $c(L/k) \leq [L:k](q-1)$ is some constant integer depending on L and k .

Proof. See [24, Theorem 8]. \square

Lemma 3.3 *Let k be a field complete with respect to a discrete valuation and of characteristic p with perfect residue class field. Let t be a uniformizing parameter. Let k' be the extension of k obtained by joining all roots of $x^p - x = a$ to it where $a = \sum_{-m}^{\infty} c_v t^v$ for some $m > 0$, $p \nmid m$ and $c_{-m} \neq 0$. Then the different of k'/k is given by*

$$\mathcal{D}(k'/k) = \mathfrak{p}^{(m+1)(p-1)},$$

where \mathfrak{p} is the only prime of k' .

Proof. See [1, Chapter 10 §4, pp. 203–205]. \square

Lemma 3.4 *Let k be a global function field. There exists a \mathbb{Z}_p -extension of k in which an infinite number of primes of k are completely decomposed.*

Proof. [7, Theorem 3]. \square

Proof of Theorem 3.1. By Lemma 3.4, there exists a \mathbb{Z}_p -extension k_∞/k in which an infinite number of primes of k are completely decomposed. Let $\{\mathcal{P}_i : i \geq 0\}$ denote the set of these primes. Take an arbitrary positive integer t . By the Chinese Remainder Theorem we can find $f_i \in A$ such that $\text{ord}_{\mathcal{P}_i}(f_j) = \delta_{ij}$ for all $1 \leq i, j \leq t$ where δ_{ij} is the Kronecker symbol. Let k'_n be the extension of k_n obtained by joining to it all roots of

$$x^p - x = \frac{1}{f_1 f_2 \cdots f_t}.$$

Clearly $k' = k'_0$ is a cyclic extension of degree p over k . By Lemma 3.3, one sees that $\mathcal{P}_1, \dots, \mathcal{P}_t$ are ramified in k'/k . Thus all the primes of k_n lying above $\mathcal{P}_1, \dots, \mathcal{P}_t$ are ramified in k'_n/k_n . The number of such primes of k_n is tp^n since $\mathcal{P}_1, \dots, \mathcal{P}_t$ are completely decomposed in k_n/k . Since all the ramification indices are equal to p , applying Lemma 3.2 one has

$$p(q-1)|J_{k'_n}^G| = h_{k_n} p^{tp^n+s} c(k'_n/k_n),$$

where $c(k'_n/k_n)$ is an integer and s is the number of other ramified primes of k'_n . Let X'_n be the p -primary subgroup of the class group of k'_n . Considering the p -divisibility of the above identity one has $p^{tp^n-1} |X'_n|$. Thus $e_n \geq tp^n - 1$ and $\mu \geq t$ for the extension k'_∞/k'_0 . Noticing that the integer t is arbitrary, we can find cyclic extension k'/k of degree p and a \mathbb{Z}_p -extension k'_∞ such that $t_1 = \mu(k'_\infty/k') > N$. Now for the same reason we can find a series of cyclic extensions $k^{(i)}/k$ ($i \geq 1$) of degree p and geometric \mathbb{Z}_p -extensions $k^{(i)}/k^{(i)}$ such that $\mu(k^{(i+1)}/k^{(i+1)}) > \mu(k^{(i)}/k^{(i)})$. This concludes our proof of Theorem 3.1. \square

4 Class numbers in \mathbb{Z}_p^d -extensions

In this section we let K_∞/k be a \mathbb{Z}_p^d -extension satisfying assumption **FRP** with Galois group Γ . Let K_n be the fixed field of Γ^{p^n} . Let p^{e_n} be the p -part

of the class number of K_n . We want to know what e_n will look like when n gets very large. For all the results in this section we only sketch the outline of the proofs.

Let L be the maximal unramified abelian p -extension of K_∞ and $X = \text{Gal}(L/K_\infty)$. Then X can be considered as a \mathbb{Z}_p -module on which Γ acts through $G = \text{Gal}(L/k)$ in a natural way since X is a normal abelian subgroup of G . Let Λ_d denote the complete group ring of $\Gamma = \langle \sigma_1, \dots, \sigma_d \rangle$ over \mathbb{Z}_p . Take $T_i = \sigma_i - 1$ for $1 \leq i \leq d$, then it is easy to see that $\Lambda_d \cong \mathbb{Z}_p[[T_1, \dots, T_d]]$. Once we have fixed the topological generators of Γ we can regard X as a $\mathbb{Z}_p[[T_1, \dots, T_d]]$ -module.

Theorem 4.1 *X is a noetherian torsion Λ_d -module.*

Proof. The proof goes by induction on d similar to the one for [8, Theorem 1]. The first step of the induction is Theorem 2.6. Note that the assumption that there are only finitely many ramified primes in K_∞/k is essential so that the argument on top of [8, p. 207] can be applied to our situation. \square

Theorem 4.2 *For $d \geq 2$, there exist integers m_0, l_0 and a real number α determined by X such that*

$$e_n = (m_0 p^n + l_0 n + \alpha) p^{(d-1)n} + O(np^{(d-2)n}).$$

Furthermore α is rational if $d = 2$.

Proof. See [22, Theorem 3.13]. For the most part the proof relies on purely module theoretic results and Theorem 4.1. No properties special to number fields are essentially used. \square

Theorem 4.3 *If X is finite as \mathbb{Z}_p -module, let $\lambda = \text{rank}(X)$. Then there exist integer ν such that*

$$e_n = \lambda n + \nu \quad \text{for all } n \gg 0.$$

Proof. See [4, Theorem 2]. \square

Theorem 4.4 *If L/k is a \mathbb{Z}_p -extension inside K_∞/k . Then $\mu(L/k) = m_0$ provided L lies outside of finitely many \mathbb{Z}_p^{d-1} -extensions of k .*

Proof. See [19, Theorem I]. □

In classical setting, Cuoco considered the relations between invariants of \mathbb{Z}_p^{d-1} -extensions inside \mathbb{Z}_p^d -extension and those of the \mathbb{Z}_p^d -extension. We can also obtain the following result

Theorem 4.5 *Let E be a \mathbb{Z}_p^{d-1} -extension of k and k_∞ a \mathbb{Z}_p -extension of k such that $K_\infty = Ek_\infty$ and $E \cap k_\infty = k$. Let k_n be the unique subfield of k_∞ of degree p^n over k . Then there are constants l , m_1 , c and c_1 , which are independent of both k_∞ and n , such that for all sufficiently large n ,*

$$m_0(Ek_n/k_n) = m_0(K_\infty/k)p^n + m_1n + c_1, \text{ and } l_0(Ek_n/k_n) = lp^n + c.$$

Furthermore, $m_1 \leq l_0(K_\infty/k)$.

Proof. The proof of this theorem is almost the same as the one in number field case under the assumption **FRP**. One need use Theorem 4.1 the proof of which depends on the assumption **FRP** crucially. For details we refer the interested readers to the proof of [3, Theorem 1]. □

Remark. The following question corresponding to [19, Theorem II] is open in function field case: Fix base field k , are the Iwasawa μ -invariants bounded for all the \mathbb{Z}_p -extensions over k ? The reason it is unsolved is that for any global function field there are infinitely many \mathbb{Z}_p -extensions over it. However, if we allow the base field to change then Theorem 3.1 shows that the μ -invariants are unbounded.

5 Iwasawa invariants over \mathbb{Z}_p^2 -extensions

Let k_∞ and k'_∞ be two \mathbb{Z}_p -extensions of k (which are constant or geometric extensions) such that $k_\infty \cap k'_\infty = k$. If we put $K_n = k_n k'_\infty$ where k_n is the n -th layer of k_∞ one sees that K_n is a \mathbb{Z}_p -extension of k_n and therefore we can speak of the Iwasawa invariants $\lambda_n = \lambda(K_n/k_n)$ and $\mu_n = \mu(K_n/k_n)$. In this section we will discuss how these invariants change when n increases. By using Rosen's results we will make the following theorem (essentially due to Cuoco) more precise when k_∞ is a constant \mathbb{Z}_p -extension of k .

Theorem 5.1 *Let $K = k_\infty k'_\infty$ and assume that there are only finitely many primes of k which are ramified in K/k . Then there are constants l, m_0, m_1, c and c_1 , which are independent of n , such that for all sufficiently large n ,*

$$\lambda_n = lp^n + c, \mu_n = m_0 p^n + m_1 n + c_1.$$

Furthermore, $m_0 = m_0(K/k)$ is independent of choices of k_∞ and k'_∞ .

Proof. This is the special case of Theorem 4.5. □

Now we will study the case when one of k_∞ and k'_∞ is a constant \mathbb{Z}_p -extension. First, we state some results from [23] and [5]. From algebro-geometric point of view k can be realized as the field of \mathbb{F}_q -rational functions of a complete non-singular curve \mathcal{C} . The Jacobian J of \mathcal{C} is an abelian variety of dimension g_k , the genus of k . The degree zero divisor class group $C_0(k)$ of k is isomorphic to $J(\mathbb{F}_q)$, the group of \mathbb{F}_q -rational points on J . Let F_n denote the extension over \mathbb{F}_q of degree p^n and $F_\infty = \bigcap_n F_n$. For any field E let $J(E)_m$ and $J(E)(p)$ denote the points of order m and p -power in $J(E)$ respectively. Put $J_p = J(\overline{\mathbb{F}_q})_p$, where $\overline{\mathbb{F}_q}$ is the algebraic closure of \mathbb{F}_q . Let $\mathbb{F}_q(J_p)$ be the field obtained by adjoining the coordinates of the points of J_p to \mathbb{F}_q .

Definition. We say that p is *regular* for k if $p \nmid [\mathbb{F}_q(J_p) : \mathbb{F}_q]$. A finite abelian p -group A is said to have type (m_1, \dots, m_r) ($m_i > 0$) if $A \cong \mathbb{Z}/(p^{m_1}) \times \dots \times \mathbb{Z}/(p^{m_r})$.

Let r_k be the p -rank of $J(F_\infty \cap \mathbb{F}_q(J_p))$. It is easy to see that if p is regular then $r_k = p\text{-rank}(J(\mathbb{F}_q)) = p\text{-rank}(C_0(k))$. Put

$$p' = \begin{cases} 4 & \text{if } p = 2, \\ p & \text{if } p > 2. \end{cases}$$

If p is irregular for k let $F_{\delta(p)} = F_\infty \cap \mathbb{F}_q(J_{p'})$. If p is regular for k let $\delta(p) = 1$ if $p = 2$ and $\delta(p) = 0$ if $p > 2$.

Lemma 5.2 *Let (m_1, \dots, m_{r_p}) be the type of $J(F_{\delta(p)})(p)$. Then the type of $J(F_j)(p)$ is $(m_1 + j - \delta(p), \dots, m_{r_p} + j - \delta(p))$ for all $j \geq \delta(p)$.*

Proof. See [23, p. 293]. □

Remark. In the above lemma $r_p \geq r_k$ and the equality holds when $p > 2$. Also r_p depends only on k , so we will write $r_p(k)$ if we want to emphasize this.

Lemma 5.3 *Let $K = k\tilde{\mathbb{F}}_q$. Let k' be a geometric extension of k of degree p and $L = k'\tilde{\mathbb{F}}_q$. Then $r_p(k)$ (resp. $r_p(k')$) is the p -rank of the class group of K (resp. L). Moreover*

$$r_p(k) - 1 = p(r_p(k') - 1) + (p - 1)t$$

where t is the number of ramified primes of L in L/K .

Proof. The first statement follows directly from Lemma 5.2 and the second is a special case of [5, (1.1)]. \square

Proposition 5.4 *The Iwasawa invariants $\lambda(k\tilde{\mathbb{F}}_q/k) = r_p$ and $\mu(k\tilde{\mathbb{F}}_q/k) = 0$.*

Proof. Let $k_n = kF_n$ be the n -th layer of the constant \mathbb{Z}_p -extension $k\tilde{\mathbb{F}}_q/k$. By Lemma 5.2, when n is sufficiently large the p -primary part of $C_0(k_n)$ has p -rank r_p and its order is increased by r_p when n is increased by 1. The proposition follows at once. \square

Now we need zeta function of k in our computation of the rank r_p . The zeta function is defined by

$$\zeta_k(s) = \prod_{\mathcal{P}: \text{prime of } k} (1 - N\mathcal{P}^{-s})^{-1},$$

where the norm $N\mathcal{P} = q^{\deg \mathcal{P}}$. Let g be genus of k . It is well known that $\zeta_k(s)$ is a rational function in $u = q^{-s}$ and can be written as

$$\zeta_k(s) = \frac{L_k(s)}{(1-u)(1-qu)}, \quad (2)$$

where $L(u) = \prod_{i=1}^{2g} (1 - w_i u)$ is a polynomial of degree $2g$ with rational integer coefficients. It is a standard fact that $L(1)$ is the class number of k . Moreover the w_i 's are algebraic integers and $w_i \rightarrow qw_i^{-1}$ is a permutation of these numbers. One very interesting fact about $L_k(u)$ is that if E/k is a constant field extension of degree m then

$$L_E(u) = \prod_{i=1}^{2g} (1 - w_i^m u).$$

Proposition 5.5 *Let $\zeta_k(s)$ be the zeta function of k as in (2). Then r_p is the number of ω_i 's such that $\omega_i \equiv 1 \pmod{\wp}$, where \wp is any fixed prime of $\mathbb{Q}(\omega_1, \dots, \omega_{2g})$ lying above $p \in \mathbb{Z}$.*

Proof. Let k_n be the constant field extension of k of degree p^n . Then the zeta function of k_n is

$$\zeta_{k_n}(s) = \prod_{i=1}^{2g} (1 - \omega_i^{p^n} u) / (1 - u)(1 - qu).$$

Hence the class number of k_n can be expressed as

$$h_n = \prod_{i=1}^{2g} (1 - \omega_i^{p^n}).$$

Let ω be one of the ω_i 's above. Then one can write $\omega = 1 + a$ for some algebraic integer a . Thus

$$\begin{aligned} \omega^{p^n} - 1 &= (1 + a)^{p^n} - 1 \\ &= a \left(C_{p^n}^1 + C_{p^n}^2 a + \cdots + C_{p^n}^{p^n-1} a^{p^n-2} + a^{p^n-1} \right). \end{aligned}$$

Let \wp be a fixed prime of $\mathbb{Q}(\omega_1, \dots, \omega^{2g})$ lying above p . If \wp does not divide $a = \omega - 1$ then \wp does not divide $\omega^{p^n} - 1$ either since $p \mid C_{p^n}^i$ for $0 < i < p^n$.

In the algebraic closure Ω_p of \mathbb{Q}_p , $\omega^{p^n} - 1 = \prod_{i=1}^{p^n} (\omega - \zeta_i)$ where the ζ_i runs through the p^n -th root of unity. If \wp divides $a = \omega - 1$, then $\text{ord}_p(a) > 1/(p^m - 1)$ for all $m > n_0 = \log_p(1 + 1/\text{ord}_p(a))$ and therefore

$$\text{ord}_p(\omega - \zeta_i) = \text{ord}_p(a + 1 - \zeta_i) = \text{ord}_p(1 - \zeta_i)$$

for all primitive m -th roots of unity. Thus the above holds with at most finitely many ($< p^{n_0+1}$) exceptions. Since $p^n = \prod_{i=1}^{p^n-1} (1 - \zeta_i)$ one sees that there is a constant c independent of n such that

$$\text{ord}_p(\omega^{p^n} - 1) = n + c \quad \text{for all } n \gg 0.$$

Let λ be the number of ω_i 's such that $\omega_i \equiv 1 \pmod{\wp}$. Then $\text{ord}_p(h_n) = \lambda n + \nu$ for some constant ν . Now we can finish the proof by combining this with Proposition 5.4. \square

Let k_∞/k be a geometric \mathbb{Z}_p -extension of k with finitely many ramified primes. Let λ_n and μ_n be the Iwasawa invariants for $k_n \tilde{\mathbb{F}}_q/k_n$. Then $\mu_n = 0$ for all n by Proposition 5.4. From Proposition 5.4 and Lemma 5.3 one has

$$\lambda_n - 1 = p(\lambda_{n-1} - 1) + (p - 1)t,$$

where t is the number of the primes of k_n ramified in k_n/k_{n-1} . Choose some n_0 such that k/k_{n_0} is totally ramified, then t is independent of n when $n \geq n_0$. By solving the recurrence formula one has :

Proposition 5.6 *Let k_∞/k be a geometric extension of k . Let λ_n and μ_n be the Iwasawa invariants of $k_n\tilde{\mathbb{F}}_q/k_n$. Then*

$$\begin{aligned}\mu_n &= 0 \quad \text{for all } n \geq 0 \\ \lambda_n &= p^{n-n_0}(\lambda_{n_0} + t - 1) - t + 1 \quad \text{for all } n \geq n_0.\end{aligned}$$

Theorem 5.7 *Let k_∞/k be a geometric \mathbb{Z}_p -extension with finitely many ramified primes. Then there exists some constant c such that for $n \gg 0$*

$$\begin{aligned}\mu(k_\infty F_n/k F_n) &= \mu(k_\infty/k) + n c, \\ \lambda(k_\infty F_n/k F_n) &= \lambda(k_\infty/k).\end{aligned}$$

Proof. Choose some m_0 such that all ramified primes in k_∞/k_{m_0} are totally ramified. For the extension k_∞/k one has

$$e_m = \lambda m + \mu p^m + \nu \quad \text{for all } m \gg 0$$

for some constants $\lambda = \lambda(k_\infty/k)$, $\mu = \mu(k_\infty/k)$ and ν . One knows that $l_m = r_p(k_m)$ and $l_{m_0} = r_p(k_{m_0})$ are the λ -invariants of $k_m\tilde{\mathbb{F}}_q/k_m$ and $k_{m_0}\tilde{\mathbb{F}}_q/k_{m_0}$ respectively by Proposition 5.4. For the extension $k_\infty F_n/k F_n$, by Lemma 5.2 one has

$$\text{ord}_p(h(K_m)) = e_m + n l_m = \lambda m + \mu p^m + \nu + n l_m \quad \text{for all } n \gg 0$$

where K_m is the m -th layer of $k_\infty F_n/k F_n$ which is a constant extension of k_m of degree p^n . Then Proposition 5.6 yields

$$l_m = p^{m-m_0}(l_{m_0} + c' - 1) - c' + 1$$

for some constant c' . Take $c = p^{-m_0}(l_{m_0} + c' - 1)$ one has

$$\text{ord}_p(h(K_m)) = \lambda m + \mu p^m + \nu + n r_n = \lambda m + (\mu + n c)p^m + \nu - c' + 1.$$

This shows that for $n \gg 0$

$$\begin{aligned}\mu(k_\infty F_n/k F_n) &= \mu(k_\infty/k) + n c, \\ \lambda(k_\infty F_n/k F_n) &= \lambda(k_\infty/k)\end{aligned}$$

as desired. □

Corollary 5.8 *In Theorem 5.1, if the \mathbb{Z}_p^2 -extension K contains a constant \mathbb{Z}_p -extension then $m_0 = 0$. If k_∞/k is constant \mathbb{Z}_p -extension then $l = 0$.*

Proof. Note that m_0 depends only on K by Theorem 5.1. □

Acknowledgement. Some of the results form part of the first author's Ph.D thesis of Brown University (1996). Both authors would like to thank Professor M. Rosen for his encouragement and help. He kindly provided the proof of Proposition 2.13. We also express our gratitude towards the referee whose suggestion makes our exposition better.

References

- [1] Artin, E., Algebraic Numbers and Algebraic Functions. Gordon and Breach Science Publishers, New York-London-Paris, 1967.
- [2] Cuoco, A., The growth of Iwasawa invariants in a family. *Comp. Math.* **41**(1980), pp. 415–417
- [3] Cuoco, A.A., Generalized Iwasawa invariants in a family, *Comp. Math.* **51**(1984), pp. 89–103
- [4] Cuoco, A., Monsky, P., Class numbers in \mathbf{Z}_p^d -extensions. *Math. Ann.* **255**(1981), pp. 235–258
- [5] D'Mello, J., Madan, M., Class group rank relations in \mathbb{Z}_ℓ -extension. *Manuscripta Math.* **41**(1983), pp. 75–107
- [6] Ferrero, B., Washington, L., The Iwasawa invariant μ vanishes for abelian number fields. *Ann. of Math.* **109**(1979), pp. 377–395
- [7] Gold, R., Kisilevsky, H., On geometric \mathbf{Z}_p -extensions of function fields. *Manuscripta Math.* **62**(1988), pp. 145–161
- [8] Greenberg, R., The Iwasawa invariants of Γ -extension of a fixed number field. *Amer. J. Math.* **95**(1973), pp. 204–214
- [9] Guo, L., Shu, L., Class numbers of cyclotomic function fields. Preprint.
- [10] Hayes, D., Explicit class field theory in global function fields. In: Studies in Algebra and Number Theory, G.C. Rota (ed.), Academic Press, 1979, pp. 173–217
- [11] Iwasawa, K., On Γ -extensions of algebraic number fields. *Bull. Amer. Math. Soc.* **65**(1959), pp. 183–226

- [12] Iwasawa, K., On \mathbf{Z}_l -extensions of algebraic number fields. *Ann. of Math.* **98**(1973), pp. 246–326
- [13] Iwasawa, K., On the μ -invariant of \mathbf{Z}_l -extensions. In: Number Theory, Algebraic Geometry and Commutative Algebra, in honor of Y. Akizuki, Kinokuniya, Tokyo, 1973, pp. 1–11
- [14] Lang, S., Cyclotomic Fields I and II. Combined second edition. Graduate Texts in Mathematics 121, Springer-Verlag, New York 1990
- [15] Li, C., \mathbf{Z}_p -extensions of function fields over finite fields. Ph.D thesis, Brown University, 1996
- [16] Li, C., Zhao, J., Class number growth of a family of \mathbf{Z}_p -Extensions over global function fields. Preprint.
- [17] Minardi, J., Iwasawa modules for \mathbf{Z}_p^d -extensions of number fields. In: Canad. Math. Soc. Conf. Proc. 7, H. Kisilevsky and J. Labute (eds.), Amer. Math. Soc. (1987), pp. 237–242
- [18] Monsky, P., On p -adic power series. *Math. Ann.* **255**(1981), pp. 217–227
- [19] Monsky, P., Some invariants of \mathbf{Z}_p^d -extensions. *Math. Ann.* **255**(1981), pp. 229–233
- [20] Monsky, P., p -ranks of class numbers in \mathbf{Z}_p^d -extensions, *Math. Ann.* **263**(1983), pp. 509–514
- [21] Monsky, P., Class numbers in \mathbf{Z}_p^d -extensions, II, III, IV, *Math. Zeit.* **191**(1986), pp. 377–395, **193**(1986), pp. 491–514, **196**(1987), pp. 547–572
- [22] Monsky, P., Fine estimates for the growth of e_n in \mathbf{Z}_p^d -extensions, in: Algebraic Number Theory—in honor of K. Iwasawa, J. Coates, R. Greenberg, B. Mazur and I. Satake (eds.), Advanced Studies in Pure Math. **17**(1989), pp. 309–330
- [23] Rosen, M., The asymptotic behavior of the class group of function field over a finite field. *Ann. of Math.* **24**(1973), pp. 287–296
- [24] Rosen, M., Ambiguous divisor classes in function fields. *J. Number Theory* **19**(1977), pp. 160–173
- [25] Rosen, M., Some remarks on the p -rank of an algebraic curve. *Arch. der Math.* (Basel) **41**(1983), pp. 143–146

- [26] Sinnott, W., On the μ -invariant of the Γ -transform of a rational function. *Inv. Math.* **75**(1984), pp. 273–282
- [27] Sinnott, W., Γ -transforms of rational function measures on \mathbb{Z}_S . *Inv. Math.* **89**(1987), pp. 139–157
- [28] Thakur, D., Iwasawa theory and cyclotomic function fields, *Contemp. Math.* **174**(1994), pp. 157–165.
- [29] Washington, L., Introduction to Cyclotomic Fields. Graduate Texts in Mathematics 83, Springer-Verlag, New York 1982

Chao-Qun Li
Jianqiang Zhao
Department of Mathematics
Brown University
Providence, RI 02912